■ Elliptical Orbits in a Gravitational Field

```
Clear["Global`*"];
```

- **Define the orbital constants and setup the initial conditions. r0 is the radius of the circular orbit. This orbit has $\omega 0 = \sqrt{GM/r0^3}$ from equating the gravitational and the centrifugal accelerations. (Note that setting $GM = r0^3$ gives $\omega 0 = 1$.) For an elliptical orbit the angular velocity is given as $\omega 0 + d\omega$ at the apogee/perigee where r[t] = r0, and the angular momentum L is calculated at this point.**

```
val = {r0 Æ 10, dw Æ .1, GM Æ 1000, m Æ 1 };

init = {r[0] ä r0 , r'[0] ä 0, q[0] ä Pi};
```

$$L = m\, r0^2 \left(\sqrt{GM/r0^3} + dw\right);$$

- **Define the equations of motion: 1st is the time derivative of the total energy* which is zero, and 2nd is just ang. momentum conservation. Numerically solve for r and θ over time range of several revolutions.**

  * Note that the second term in the kinetic energy, $\frac{m r^2 w^2}{2}$, can be rewritten as $\frac{L^2}{2 m r^2}$.
  Also the energy time derivative equation has a common factor r'[t] divided out.

$$eq = \left\{ \frac{1}{r'[t]}\, \partial_t \left(\frac{m\, r'[t]^2}{2} + \frac{L^2}{2\, m\, r[t]^2} - \frac{GM\, m}{r[t]} \right) \text{ ä } 0,\ m\, r[t]^2\, q'[t] \text{ ä } L \right\} \text{ // Simplify;}$$

```
dsol = NDSolve[Join[eq, init] /. val, {r[t], q[t]},
            {t, 0, 25}][[1]];
```

- **Make a plot of the elliptical (red) and circular (blue) orbits, which have the same angular momentum but different energies. Note that the orbits close after one revolution.**

```
ParametricPlot[
  Evaluate[{{r[t] Cos[q[t]], r[t] Sin [q[t]]} /. dsol, {r0 Cos[t], r0 Sin[t]} /. val}],
  {t, 0, 25}, PlotStyle Æ {RGBColor[1, 0, 0], RGBColor[0, 0, 1]}, AspectRatio Æ Automatic];
```