■ Elliptical Orbits in a Gravitational Field

```
Clear["Global`*"];
```

- **Define the orbital constants and setup the initial conditions. r0 is the circular orbit radius. The constant dr defines an elliptical orbit with its apogee given by r0 + dr. All orbits are assumed to have the same angular momentum L = $m\, r0^2\, \omega 0 = m \sqrt{GM\, r0}$. Note that setting GM = $r0^3$ sets the circular orbit angular velocity $\omega 0 = 1$.**

```
val = {r0 Æ 10, dr Æ 5, GM Æ 1000, m Æ 1 };

init = {r[0] ä r0 + dr, r'[0] ä 0, q[0] == 0};
```

- **Define the equations of motion: 1st is the time derivative of the total energy* which is zero, and 2nd is just the angular momentum. Numerically solve for r and θ over time range of several revolutions.**

* Note that the second term in the kinetic energy, $\frac{m r^2 \omega^2}{2}$, can be rewritten as $\frac{L^2}{2 m r^2} = \frac{GM m r0}{2 r^2}$.
Also the energy time derivative equation has a common factor r'[t] divided out.

```
eq = { 1/r'[t] ∂t ( m r'[t]^2/2 + GM m r0/(2 r[t]^2) - GM m/r[t] ) ä 0, m r[t]^2 q'[t] ä m Sqrt[GM r0] } // Simplify;
```

```
dsol = NDSolve[Join[eq, init] /. val, {r[t], q[t]},
          {t, 0, 25}][[1]];
```

- **Make a plot of the elliptical (red) and circular (blue) orbits, which have the same angular momentum but different energies. Note that the orbits close after one revolution.**

```
ParametricPlot[
  Evaluate[{{r[t] Cos[q[t]], r[t] Sin [q[t]]} /. dsol, {r0 Cos[t], r0 Sin[t]} /. val}],
  {t, 0, 25}, PlotStyle Æ {RGBColor[1, 0, 0], RGBColor[0, 0, 1]}, AspectRatio Æ Automatic];
```