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Evolutionary dynamics of biological auctions

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ABSTRACT

Many scenarios in the living world, where individual organisms compete for winning positions (or resources), have properties of auctions. Here we study the evolution of bids in biological auctions. For each auction, n individuals are drawn at random from a population of size N. Each individual makes a bid which entails a cost. The winner obtains a benefit of a certain value. Costs and benefits are translated into reproductive success (fitness). Therefore, successful bidding strategies spread in the population. We compare two types of auctions. In "biological all-pay auctions", the costs are the bid for every participating individual. In "biological second price all-pay auctions", the cost for everyone other than the winner is the bid, but the cost for the winner is the second highest bid. Second price all-pay auctions are generalizations of the "war of attrition" introduced by Maynard Smith. We study evolutionary dynamics in both types of auctions. We calculate pairwise invasion plots and evolutionarily stable distributions over the continuous strategy space. We find that the average bid in second price all-pay auctions is higher than in all-pay auctions, but the average cost for the winner is similar in both auctions. In both cases, the average bid is a declining function of the number of participants, n. The more individuals participate in an auction the smaller is the chance of winning, and thus expensive bids must be avoided.

1. Introduction

In biology, it is common for organisms to compete for resources and incur some costs for the competition. It has been shown that these competitions often have the structure of an auction (Rose, 1978; Haigh and Rose, 1980). The most popular example is the *war of attrition* as a model of conflicting animals where the time an individual is willing to stay in the conflict corresponds to the bid in an auction (Maynard Smith, 1974, 1982; Bishop et al., 1978). In classical auctions, there are a number of participants bidding for an item, the highest bidder is the winner and either pays the highest bid (first-price auctions Vickrey, 1961; Cassady, 1980; Krishna, 2009) or the second highest bid (second-price auctions Vickrey, 1961; Clarke, 1971; Groves, 1973; Krishna, 2009). In this paper, we consider biological auctions, where all the participants of the auctions have to pay, in contrast to classical auctions, where only the winner pays (Baye et al., 1996; Krishna and Morgan, 1997). We consider two types of biological auctions: (1) *biological all-pay auctions* (for short, all-pay auctions), where every participant pays their own bid, and (2) *biological second price all-pay auctions* (for short, second price all-pay auctions), where every participant other than the winner pays their own bid, and the winner pays the second highest bid.

The model of all-pay auctions is a generalization of the *scotch auction* from 2 to n players (Rose, 1978; Haigh and Rose, 1980). An example for the scotch auction is the growth competition among plants. The investment in growth corresponds to the bid in such a competition, whereas the additional sunlight can be considered as the benefit of winning this auction. Similarly, the model of second price all-pay auctions is a generalization of the war of attrition from 2 to n players (Maynard Smith, 1974, 1982). This model is also discussed in (Haigh and Cannings, 1989). In this paper, we study the evolutionary dynamics of all-pay auctions and second price all-pay auctions, and also the evolutionarily stable strategies in such auctions.

In economics, the theory of auctions and in particular, all-pay auctions have been extensively studied (Weber, 1985; Bulow and Roberts, 1989; Amann and Leininger, 1996; Baye et al., 1996; Krishna and Morgan, 1997; Siegel, 2009; Krishna, 2009), along with its applications in lobbying, rent seeking, etc. (Becker, 1983; Moulin, 1986; Baye et al., 1993; Anderson et al., 1998; Bulow and Klemperer, 1999). However, none of these works considers the evolutionary dynamics of such auctions and the properties of evolutionarily stable strategies. The economics literature considers all-pay auctions in the more general setting of incomplete information and different values for each participant,

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and studies different notions of symmetric and asymmetric equilibria (Baye et al., 1996; Siegel, 2009). All-pay auctions with complete information are analyzed in Baye et al. (1996), where the mixed equilibrium for the case of 3-participants is calculated as an example. We present the mixed equilibrium for the general case of n-participants and also study whether the mixed equilibrium is an ESS.

Our approach is different as we explore the evolutionary dynamics of these auctions which are driven by a selection–mutation process in a population of individuals. Our main results are as follows.

(a) A strategy for a player is a recipe to bid. Formally, a strategy is a probability distribution over the possible values of the bid. Both for all-pay auctions and second price all-pay auctions, we derive the strategy (i.e., the probability distribution function) for n participants that is a mixed equilibrium. For second price all-pay auctions, we can show using the results of Haigh and Cannings (1989) that the mixed equilibrium strategy that we obtain is an evolutionarily stable strategy (ESS) for all $n \geq 2$. In contrast, we show that for all-pay auctions the mixed equilibrium is not an ESS for $n = 2$, but is an ESS for all $n > 2$. Our analytical results are independent of any assumption on the population size.

(b) We present the pairwise invasion plots when there are two strategies. An interesting finding in our setting (which to the best of our knowledge has no correspondence in economics) is that as the number of participants n is large, then any strategy is invadable by any other strategy.

(c) We study the evolutionary dynamics of both types of auctions with a computer simulation for a finite population size where the players can choose only of finitely many pure strategies. There is no static equilibrium in the case of a finite population with pure strategies and the evolutionary dynamics show stochastic fluctuations. The cyclic behavior within the stochastic fluctuations is as follows: given a population of low bidding strategies, the selection–mutation process shifts the population slowly towards higher bidding strategies, but then in the population of very high bidding strategies the selection–mutation process shifts the population towards very low bidding strategies as the high bidders incur high costs when they participate in the same auctions and therefore the low bidders can invade again. However very interestingly, even though there is no equilibrium distribution and there is cyclic behavior, there is a close correspondence of the time-average distribution of individuals over the strategy space and the theoretical ESS distribution. In every generation, for every strategy value, we consider the number of individuals that adopt the strategy, and then over a long period of generations, we obtain the time-average of the number of individuals who adopted a strategy. This gives us the time-average distribution of individuals over the strategy space, and this average distribution converges to the analytical ESS distribution. As we have a finite population and the individuals can only play pure strategies, this convergence result was not necessarily to expect. If the population size is large, then we have excellent convergence, whereas if the population size is small, the convergence is less accurate. The study of the evolutionary dynamics, the cyclic behavior, and the convergence of the average distribution to the analytical ESS are the key findings of the paper.

Evolutionary game theory and the notion of evolutionarily stable strategies (ESS) was introduced by Maynard Smith and Price (1973). The stability of strategies and their dynamics have been studied in many different evolutionary games (Taylor and Jonker, 1978; Hofbauer and Sigmund, 1988; Foster and Young, 1990; Friedman, 1991; Cressman, 1992; Nowak and Sigmund, 1992, 1993; Crawford, 1995; Weibull, 1995; Samuelson, 1998; Hofbauer, 2000; Nowak and Sigmund, 2004; Imhof et al., 2005; Nowak, 2006; Gintis, 2009). For stochastic evolutionary game dynamics in finite sized populations, see (Nowak et al., 2004; Imhof and Nowak, 2006; Ohtsuki et al., 2007; Traulsen et al., 2007; Fudenberg and Imhof, 2008) and many others. Rose (1978) analyzed the 2-player ESS of scotch auctions which are a special case of all-pay auctions. Haigh and Rose (1980) extended their theory also to the war of attrition. Riley, Vickers and Cannings introduced stronger notions of ESS as the former definition holds only for infinitely large populations and for an infinitesimally number of invaders (Riley, 1979, 1980; Vickers and Cannings, 1987). Oechssler and Riedel (2001) extended the former studies on evolutionary dynamics with finite strategy sets to infinite strategy sets.

The war of attrition was introduced by Maynard Smith (1974, 1982) and has been used to model competitions between conflicting animals (Bishop et al., 1978; Maynard Smith, 1982), struggling firms (Fudenberg and Tirole, 1986) and forming coalitions in politics (Bulow and Klemperer, 1999). As we already mentioned above, the results of war of attrition can be obtained as a special case for $n = 2$ in the model of second price all-pay auctions. Bishop and Cannings (1978) analyzed a generalized war of attrition to allow different reward and cost functions, and restrict the length of the contest, Haigh and Cannings (1989) analyzed our and other n-player models of the war of attrition. In this work, we present the solution for both multi-player biological auctions, the generalization of scotch auctions through all-pay auctions and the generalization of war of attrition through second price all-pay auctions.

2. Introduction to the biological auctions model

In this section, we introduce the basic model of *biological auctions*. We consider a well-mixed population where the individuals are chosen uniformly at random for participating in auctions. The strategy of an individual is the bid for the auction. The winner of an auction is the individual with the highest bid (i.e., the individual with the highest strategy value participating in this auction). If there is a tie, then the tie is broken by a random choice (i.e., the winner is selected uniformly at random from the participants with the highest bid). The costs for participating in an auction depend on the type of the auction. According to the cost for participating in the auctions, we will distinguish between two types of biological auctions: (1) *biological all-pay auctions* (for short, all-pay auctions (APA)) and (2) *biological second price all-pay auctions* (for short, second price all-pay auctions (SAPA)). Each individual has a fitness value which is updated according to the benefit for the wins and the costs incurred for participating in the auctions. Afterward, the next generation is created where the strategy values of the new individuals are chosen from the last generation proportional to their fitness values.

3. Biological all-pay auctions

We now explain the model of biological all-pay auctions (APA) in detail. We consider a large well-mixed population of size N. Before every update of a generation to the next, there are K auctions. In each auction, a set of $n \leq N$ players (or individuals) are chosen to participate in the auction, and the players are chosen uniformly at random from the population. Each individual i has a fixed strategy value s_i ($s_i \in [0, \infty)$) and a fitness value f_i ($f_i \geq 0$). The strategy value s_i of a player is the bid of the player in an auction. The winner in an auction is the individual with the highest bid, and the winner receives a benefit of value $V \geq 0$. All the participating individuals have to pay some costs depending on the type of the auction. In APA, the costs are the bid for every participating individual. If there is a draw in an auction, then the winner is selected uniformly at random from the set of the players with the highest bid. At the end of a generation, the background fitness of the players is updated according to the wins and costs of participating in the auctions. The new generation of the same size N is created. All new individuals start with the same (equal) background fitness value but their strategy values are chosen from the individuals in the last generation proportional to the updated fitness values. In each of the K auctions in a generation, we have the following steps.

(a) A set of n individuals is selected uniformly at random from the population for each auction.
(b) Each individual selected for the auction pays the price of the strategy (or the bid) of the individual.
(c) The winner is chosen uniformly from the set of individuals with the highest bid, and the winner gets a benefit of value V.

For an individual i, with strategy value s_i, if the individual was chosen in k_i auctions, and was the winner in w_i auctions, then the updated fitness value is the sum of the background fitness value and $(w_i \cdot V - s_i \cdot k_i)$. In the next generation, an individual i chooses a strategy s_j of individual j with probability proportional to the fitness of j in the last generation. In the new generation, every individual is again assigned the same background fitness value. In the special case of $n = 2$, we have the following payoff matrix

	Player 1	Player 2
$s_1 > s_2$	$V - s_1$	$-s_2$
$s_1 = s_2$	$\frac{V}{2} - s_1$	$\frac{V}{2} - s_1$
$s_1 < s_2$	$-s_1$	$V - s_2$.

In the next two subsections, we will analyze the following two cases: (1) the analysis when there are only two strategies s and s' present in the population, and (2) the analysis when there is a mixed ESS within the continuum of strategies.

We now present two examples for APA.

Example 1. A well known phenomenon in biology is the evolution of traits of males to increase their sexual attractiveness to females even when this trait decreases their own fitness in terms of increased risk or invested energy (Darwin, 1874; Zahavi, 1975; Andersson, 1994). Examples for such traits are the tail length of widowbirds (Andersson, 1982) and tail ornaments of swallows (Møller, 1988). We now argue how this phenomenon can be modeled as APA. In general, the investment of males in such traits, which is an advertisement to increase their mating success, corresponds to the bid in APA (recall that the costs for each individual are equal to their bid). All participants have exactly the costs of their investment and the winner of the contest is the male with the highest investment (i.e., for widowbirds the male with the longest tail). The winner gets the female, which corresponds to the benefit value of the auction. These bids are rather lifetime investments than individual investments in each competition. Our analytical results still hold as the lifetime investment can be split across the participated auctions of an individual and the number of participations is not influenced by the player according to our model.

Example 2. A more specific example is courtship feeding. Again males are advertising for females where courtship feeding can be considered as an investment to increase their attractiveness to future mating partners (Helfenstein et al., 2003; Gwynne, 1984). As in Example 1, the possible benefit for the investment is the mating with the female. In contrast to the lifetime investments in Example 1, in this example the advertising costs occur in each competition for all participants. Note that each individual has to make strategic decisions to find the right trade-off of investment in advertisement and expected benefits, which is typical for an all-pay auction.

3.1. The two strategy case

We take two different strategies namely s_1 and s_2, and without loss of generality we assume $s_1 < s_2$. Let x be the frequency of individuals with the strategy s_1; we refer to them as s_1 strategists. Thus, $1 - x$ is the frequency of s_2 strategists in the population. In other words, x denotes the frequency of individuals with the lower strategy value.

Lemma 1. *The expected payoff for the s_1 strategists is*

$$p_{\text{APA}}(s_1) = \frac{x^{n-1} \cdot V}{n} - s_1. \tag{1}$$

The expected payoff for the s_2 strategists is

$$p_{\text{APA}}(s_2) = \frac{V \cdot (1 - x^n)}{n \cdot (1 - x)} - s_2. \tag{2}$$

The winner. We analyze the general case where a given strategy s is beaten by another strategy s' (i.e., s' gets higher payoff).

(a) *Case* 1. If $s < s'$ and $p_{\text{APA}}(s) < p_{\text{APA}}(s')$, then we require the following condition:

$$\frac{V \cdot x^{n-1}}{n} - s < \frac{V \cdot (1 - x^n)}{n \cdot (1 - x)} - s'$$

$$s' - s < V \cdot \left(\frac{1 - x^n}{n \cdot (1 - x)} - \frac{x^{n-1}}{n}\right) = \frac{V \cdot (1 - x^{n-1})}{n \cdot (1 - x)}. \tag{3}$$

(b) *Case* 2. If $s > s'$ and $p_{\text{APA}}(s) < p_{\text{APA}}(s')$, then we require the following condition:

$$s - s' > \frac{V \cdot (1 - x^{n-1})}{n \cdot (1 - x)}. \tag{4}$$

Invasion. We now consider the class of strategies that can invade a strategy s, i.e., we analyze the class of strategies s' that obtains a higher expected payoff as compared to s when the frequency of s individuals is $1 - \epsilon$, for ϵ close to 0. We first consider the case when $s < s'$:

$$s' - s < V \cdot \left(\frac{1 - (1 - \epsilon)^{n-1}}{n \cdot \epsilon}\right) \approx V \cdot \left(\frac{1 - (1 - \epsilon \cdot (n-1))}{n \cdot \epsilon}\right)$$
$$= V \cdot \left(1 - \frac{1}{n}\right);$$

since we can ignore higher order terms of ϵ, we have

$$(1 - \epsilon)^{n-1} \approx 1 - \epsilon \cdot (n - 1)$$

as approximation in our above equation. We now consider the case when $s' < s$; in this case we have $x = \epsilon$ as the frequency of the s' strategists, the difference between the strategy values is approximated

$$s - s' > V \cdot \left(\frac{1 - \epsilon^{n-1}}{n \cdot (1 - \epsilon)}\right) \approx \frac{V}{n}. \tag{5}$$

Theorem 1. *In* APA, *a population of s strategists is invadable by s' strategists iff*

$$s' \in \left(0, \max\left(0, s - \frac{V}{n}\right)\right) \cup \left(s, \min\left(V, V \cdot \left(1 - \frac{1}{n}\right) + s\right)\right).$$

Note that we only need to consider strategies of value up to V. The reason is as follows: for a strategy s of value V' such that $V' > V$, let $V' - V = \varepsilon > 0$. Then any strategy $0 \le s' < \varepsilon$ always dominates V', as the best possible payoff for s is $-\varepsilon$

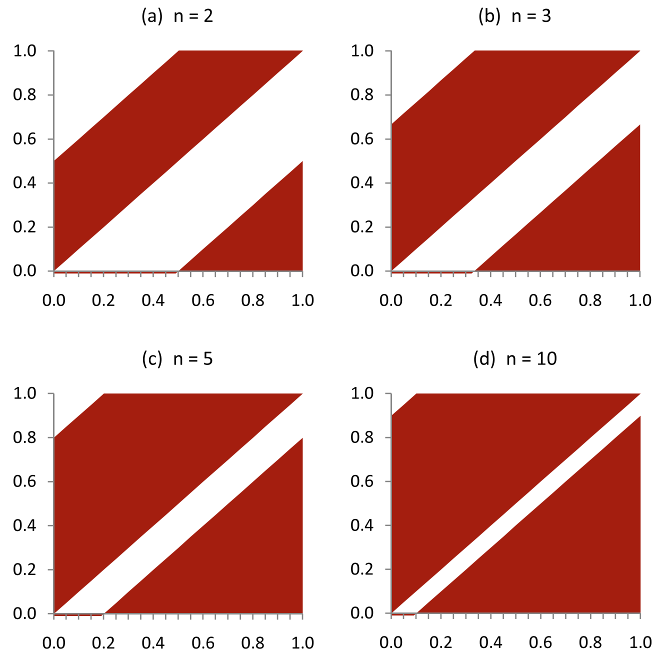

Fig. 1. The figure shows the invasion regions for APA when there are two strategies only, for $n = 2, 3, 5$, and 10. The invasion region is shown as the shaded region, and the regions are derived from Eq. (5). The regions for SAPA are similar and the difference is instead of the unit box they extend from 0 to ∞. The figure is interpreted as follows: for any strategy value v, the shaded region indicates the strategy values that can invade a population that consists of only v strategists. For example, if the whole population consists only of 0.1 strategists, then invaders have to bid more than 0.1 and less than 0.6 for a successful invasion when $n = 2$. If their strategy value is higher than 0.6 or lower than 0.1, then they cannot take over the whole population. Fig. 1(a)–(d) show the results for $n = 2, 3, 5$, and 10, respectively.

(in any environment) and the worst possible payoff for s' is less than $-\varepsilon$ (in any environment). In Fig. 1, we show the shaded invasion regions for the two strategy case in APA when $V = 1$: the shaded regions show, in the case of only two possible strategies, the set of strategies that can defeat (or invade) a strategy. As n goes to ∞, the shaded region is the unit box.

3.2. Random strategies: Mixed ESS

Analysis of mixed ESS. It follows from the results of the previous subsection, that there is no pure ESS: suppose all the individuals play a strategy m, and then the average payoff is $\frac{V}{n} - m$, and a mutant playing $m + \delta m$, for small δm, could invade the population. Hence, if there is an ESS it must be a mixed one. Let I be a strategy defined by the probability density function $p(x)$. To obtain $p(x)$, we use Bishop–Cannings theorem, and obtain that for a strategy s in the support of I, we have $E(s, I)$ is constant, where $E(s, I)$ is the expected payoff. In other words, one necessary condition of ESS suggests that we must have $\frac{\delta E(s,I)}{\delta s} = 0$. We have

$$E(s, I) = V \cdot \left(\int_0^s p(x)dx \right)^{n-1} - s \quad (6)$$

where $p(x)$ is the probability that the strategy value is between $x + \delta x$ and V is the benefit value.

Theorem 2. *The following probability density function is a mixed equilibrium in* APA

$$p(x) = \left(\frac{1}{V} \right)^{\frac{1}{n-1}} \cdot \frac{1}{n-1} \cdot x^{\frac{2-n}{n-1}}. \quad (7)$$

The mixed equilibrium is not an ESS for $n = 2$ and is an ESS for all $n > 2$.

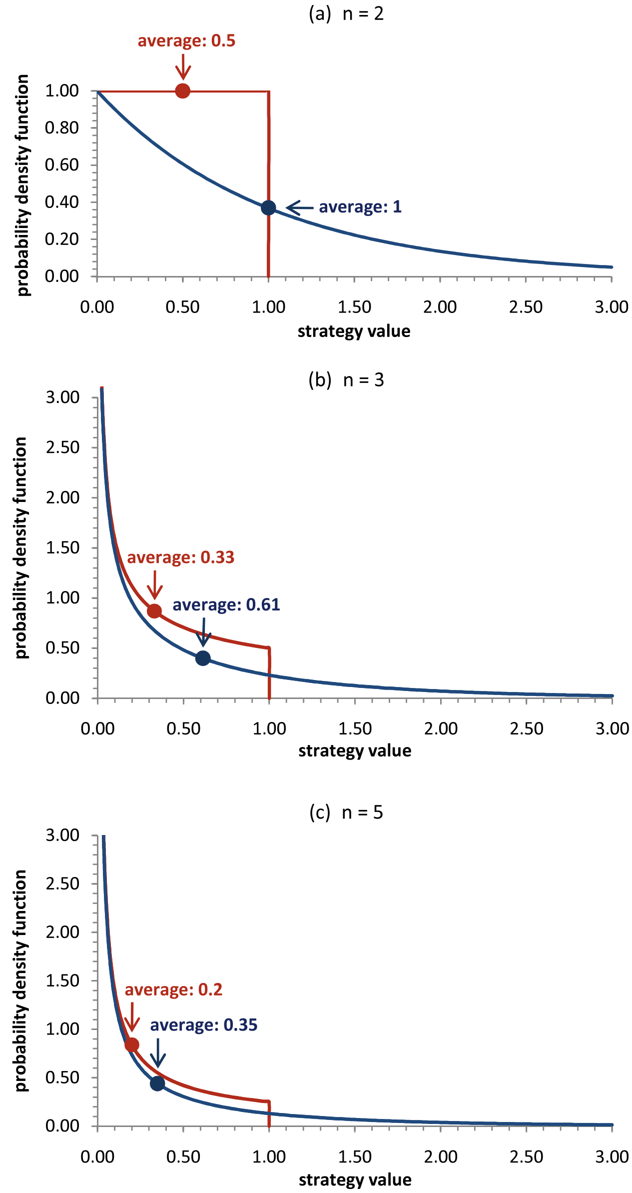

Fig. 2. The figure compares the mixed ESS for APA and SAPA for $n = 2$, $n = 3$, and $n = 5$, respectively. The result for APA is shown as the red curve and for SAPA as the blue curve. The curves are derived from Theorems 2 and 4, respectively. The dots in the figure indicate the average bid of the auction. We observe that as n increases the average bid decreases, i.e., the more participants, the less is the value of the bid in the auction. We also observe that the average bid of APA is lower than SAPA. Fig. 2(a)–(c) compares the mixed ESS for $n = 2, 3$, and 5, respectively. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

In the special case of $n = 2$, we get

$$p(x) = \frac{1}{V} \quad (8)$$

which means that our strategy values are uniformly distributed in the support. The technical proofs are given in Appendices. Fig. 2 shows the mixed equilibrium curves for $n = 2, 3$, and 5. Note that we do not need an assumption on the population size.

4. Biological second price all-pay auctions

The biological second price all-pay auction (SAPA) is similar to APA, the only difference is in the cost for the winner. In SAPA, the cost for everyone other than the winner is the cost of the bid, for the winner the cost is the second highest bid. The case of APA is equivalent to classical all-pay auctions, whereas SAPA is inspired by the idea of VCG auctions (Vickrey, 1961; Clarke, 1971; Groves, 1973).

The payoffs if $n = 2$ are

	Player 1	Player 2
$s_1 > s_2$	$V - s_2$	$-s_2$
$s_1 = s_2$	$\frac{V}{2} - s_1$	$\frac{V}{2} - s_1$
$s_1 < s_2$	$-s_1$	$V - s_1$.

From the above matrix, we observe the following interesting connection: the modification of the general theory of auctions by the VCG auction idea in the setting of all-pay auctions gives us a natural generalization of the war of attrition (Maynard Smith, 1974, 1982). Observe that in the special case of $n = 2$, SAPA gives us the payoff matrix of war of attrition. Thus SAPA (with $n \geq 2$) gives a natural generalization of the war of attrition.

Example 3. As already pointed out above, SAPA are a natural generalization of the war of attrition (Maynard Smith, 1974, 1982). We can consider the time an individual is willing to stay in the conflict as its bid. In contrast to APA, the bid of an individual is not equivalent to the costs in an auction, since a conflict ends when the second last individual is giving up within the conflict. Clearly, the costs for all participants are equivalent to their bid (i.e. the time they were willing to stay in the contest), expect the winning individual which only pays the second highest bid (i.e. the time when the second last individual left the contest). The payoff for winning the contest is typically some resource. Hence, such contests are modeled as SAPA. It was already observed in Haigh and Rose (1980) that such a contest is equivalent to a second price all-pay auction (Krishna and Morgan, 1997).

As in the case of APA, we will analyze the two strategy case and then the randomized strategy case for mixed ESS.

4.1. The two strategy case

We consider two different strategies s_1 and s_2 (WLOG, $s_1 < s_2$) and x is the frequency of s_1 strategists in the population.

Lemma 2. *The expected payoffs for the s_1 strategists are*

$$p_{2^{nd}}(s_1) = p_{APA}(s_1) = \frac{V \cdot x^{n-1}}{n} - s_1. \tag{9}$$

The expected payoffs for the s_2 strategists are

$$p_{2^{nd}}(s_2) = \frac{V \cdot (1 - x^n)}{n \cdot (1 - x)} - s_2 + x^{n-1} \cdot (s_2 - s_1). \tag{10}$$

The winner. We analyze again the general case where a given strategy s is beaten by another strategy s'.

(a) *Case* 1. If $s < s'$ and $p_{2^{nd}}(s) < p_{2^{nd}}(s')$, then we require the following condition (proof in Appendices):

$$s' - s < \frac{V}{n \cdot (1 - x)}. \tag{11}$$

(b) *Case* 2. If $s > s'$ and $p_{2^{nd}}(s) < p_{2^{nd}}(s')$, then we require the following condition:

$$s - s' > \frac{V \cdot (1 - x^{n-1})}{n \cdot (1 - x)}. \tag{12}$$

Invasion. Once again we consider the case where individuals with a strategy s' invade a population of s strategists with $s < s'$ where $x = (1 - \epsilon)$ as the frequency of s strategists, we get

$$s' - s < \frac{V}{n \cdot \epsilon} \tag{13}$$

and as ϵ is negligible it follows s is beaten by s'. If $s > s'$ where $x = \epsilon$ as the frequency of the s' strategists, we get

$$s - s' > \frac{V}{n \cdot (1 - \epsilon)} \approx \frac{V}{n}. \tag{14}$$

Theorem 3. *In* SAPA, *a population of s strategists is invadable by s' strategists iff*

$$s' \in \left(0, \max\left(0, s - \frac{V}{n}\right)\right) \cup (s, \infty).$$

The shaded regions for APA and SAPA are similar (see Fig. 1); the only difference is that they extend to ∞ in the case of SAPA. As n goes to ∞, the shaded region covers everything.

4.2. Random strategies: mixed ESS

As in the case of APA, there is no pure ESS, and we study the mixed ESS. Our goal is to obtain $p(x)$ such that $E(s, I)$ is constant. For the calculation of $E(s, I)$, we consider two cases.

(a) *Not winner:*
The probability that all other $n - 1$ individuals bid less than s is $\left(\int_0^s p(x)dx\right)^{n-1}$. Thus, the probability that at least one bid is greater is 1 − the probability that all bid less. Thus the payment is

$$s \cdot \left[1 - \left(\int_0^s p(x)dx\right)^{n-1}\right].$$

(b) *Winner:*
Since we already know the probability that all others bid less, the expected benefit is

$$V \cdot \left(\int_0^s p(x)dx\right)^{n-1}.$$

We have the following result.

Theorem 4. *The following probability density function is an ESS for all $n \geq 2$ in* SAPA

$$p(x) = \frac{1}{(n-1) \cdot V} \cdot \left(1 - \exp\left(-\frac{x}{V}\right)\right)^{\frac{2-n}{n-1}} \cdot \exp\left(-\frac{x}{V}\right). \tag{15}$$

The above result can also be obtained from (Haigh and Cannings, 1989): the distribution function was given and the analysis of ESS was studied in Haigh and Cannings (1989). We present in Appendices details of how to derive the probability density function. In the special case with $n = 2$, we have $p(x) = \frac{1}{V} \cdot \exp(-\frac{x}{V})$. Thus our result is a generalization of the result of Maynard Smith for war of attrition (Maynard Smith, 1974, 1982), and as a special case of our result (with $n = 2$), we obtain the result of war of attrition. Fig. 2 shows the mixed equilibrium curves for $n = 2, 3$, and 5; Fig. 2 also shows the mixed equilibrium curve comparison for APA and SAPA for $n = 2, 3$, and 5.

Observation. Let us consider the fraction of individuals that have a maximum strategy value of x (i.e., bid within x), and we consider the special case with $V = 1$. For APA, this is $x^{\frac{1}{n-1}}$ and for SAPA this is $(1 - \exp(-x))^{\frac{1}{n-1}}$. As n increases the number increases as well. Tables 1 and 2 tabulate the results for APA and SAPA, respectively. The description of the table is as follows: every row represents the result for a given number of participants chosen for the auction,

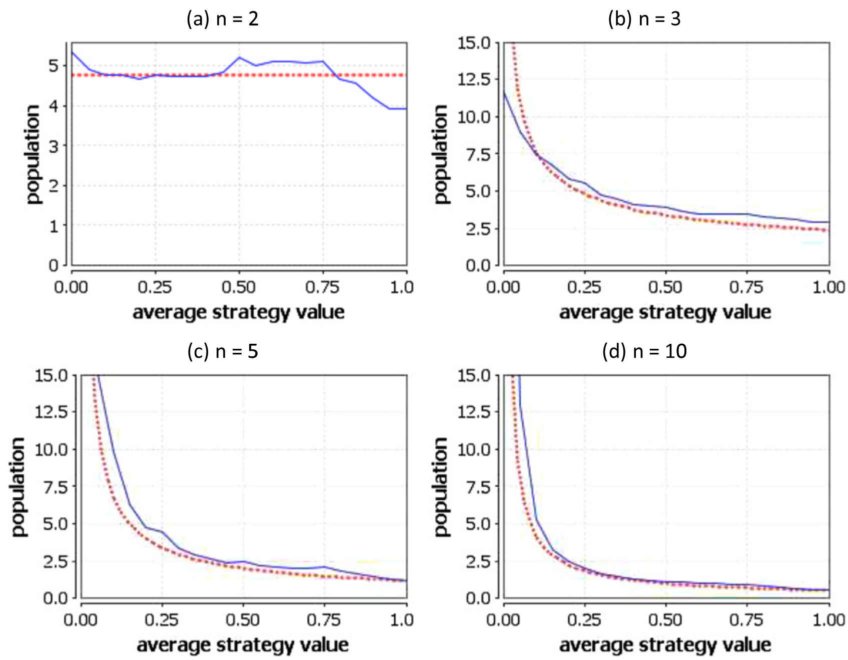

Fig. 3. The figure shows the average strategy distribution of APA after 10^7 generations obtained from our simulation results, overlayed with the ESS curve obtained from Theorem 2. The computer simulation settings are a bit different as we have only 21 different strategies (instead of a continuum of strategies) and a finite population. Nevertheless, we observe that the average strategy distribution we obtain from the simulation results is very similar to the predicted curve derived from Theorem 2. Our simulation results also show that the higher the value of n, the lower is the average strategy. Fig. 3(a)–(d) show the results for $n = 2, 3, 5$, and 10, respectively.

Table 1
APA.

	0.05	0.1	0.2
$n = 2$	0.05	0.1	0.2
$n = 3$	0.223	0.316	0.447
$n = 5$	0.472	0.562	0.668
$n = 10$	0.716	0.774	0.836

Table 2
SAPA.

	0.05	0.1	0.2
$n = 2$	0.048	0.095	0.181
$n = 3$	0.220	0.308	0.425
$n = 5$	0.469	0.555	0.652
$n = 10$	0.714	0.770	0.827

and a table entry represents the fraction of participants bidding below the corresponding value x, for example, for $n = 3$ and $x = 0.1$ in APA the answer is 0.316, i.e., 31.6% of the population bids below the value 0.1. If cooperation is interpreted as bidding low (i.e. having a small strategy value), then there is good cooperation.

5. Evolutionary dynamics and simulation results

We study the evolutionary dynamics of the auctions in a finite population with the aid of a computer simulation. We now describe our implementation and the simulation results. For the stochastic computer simulation, we fixed the benefit value V to 1. We implemented a program that takes as input the following parameters: (i) the size of the population (N), (ii) the number of auctions (K), (iii) the number of different strategies for an interval of length 1 (ℓ), (iv) the number of participants in every auction (n), (v) the mutation rate (u), and (vi) the selection intensity (δ). The strategy space is restricted to the interval [0, 1] for APA and [0, 10] for SAPA, and given the input $\ell \geq 2$ for different strategies, we consider ℓ different strategies $0, \frac{1}{\ell-1}, \frac{2}{\ell-1}, \ldots, \frac{\ell-2}{\ell-1}, 1$ for APA (and similarly for SAPA from 0 to 10). After all K auctions have been executed in a generation, the fitness values of the individuals get computed. Instead of adding the sum of the obtained benefits and costs of each individual in each generation directly to the background fitness, we multiply this sum with the selection intensity δ and then add it to the background fitness. For all our simulation results, we set $\delta = 1$ as the selection is already weak because of our high background fitness. Whenever the next generation of individuals is created, a fraction according to the chosen mutation rate u is assigned a random strategy. This makes sure that the computer simulation of the population does not get stuck in any random pure strategy. The choice of the auction can be made as APA or SAPA, and then the program simulates the auction and generates the population distribution from one generation to the next, and also produces the average distribution. Hence, we have a stochastic computer simulation of APA and SAPA on a finite population.

Average population distribution. We show the simulation results for $N = 100$ (population size), with $K = 100$ (number of auctions), $n = 2, 3, 5$, and 10 (number of participants), and for a mutation rate of 0.5%. We choose $\ell = 21$, and hence the possible strategy values are $0, 0.05, 0.1, \ldots, 0.95, 1$ for APA and $0, 0.05, 0.1, \ldots, 9.95, 10$ for SAPA, respectively. Our results are shown in Figs. 3 and 4. For APA, for the average distribution, we overlay the simulation results with the curves of the analytical results (as predicted by the ESS distribution), and observe that the simulated average distribution almost converges to the predicted ESS. The ESS curves are shown as red dotted lines. We have also observed in the simulation results that if we increase the population size, then the simulation result curves become even closer to the curves predicted by the ESS. For SAPA, our observation is similar. However, for a small population size, $N = 10$, the convergence of the simulation results to the ESS curves is less accurate. As the number of participants is increased the average distribution curve shifts towards 0.

Evolutionary dynamics. The evolutionary dynamics are illustrated by taking the population distribution at different snapshots, and this is shown in Fig. 5 for APA and in Fig. 6 for SAPA with

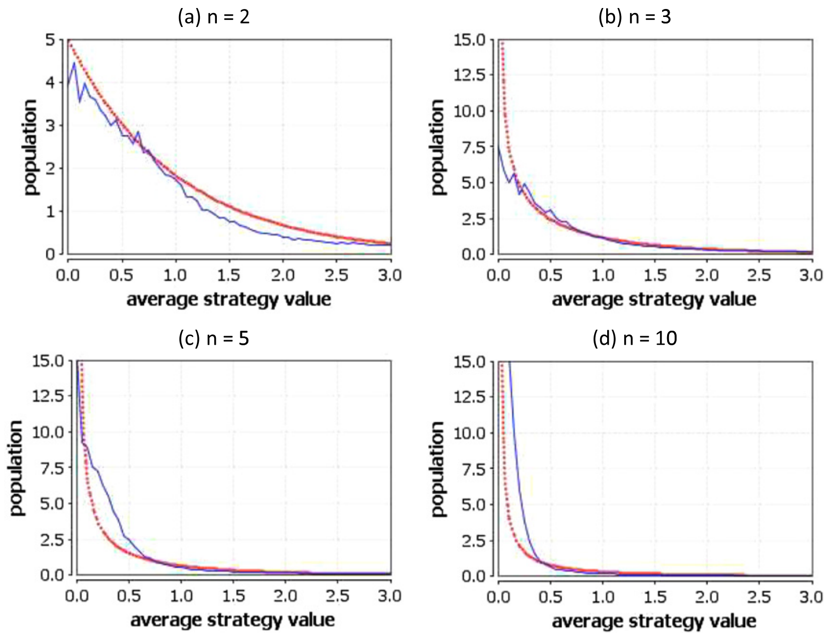

Fig. 4. The figure shows the average strategy distribution of SAPA after 10^7 generations obtained from our simulation results, overlayed with the ESS curve obtained from Theorem 4. The simulation settings are different from the infinite population model setting as described for Fig. 3. But we have 201 strategies in the interval [0, 10] such that we get similar to APA 21 strategies in the interval [0, 1]. Again we observe that the average strategy distribution we obtain from the simulation results shows excellent agreement to the predicted curve derived from Theorem 4. Fig. 4(a)–(d) show the results for $n = 2, 3, 5$, and 10, respectively.

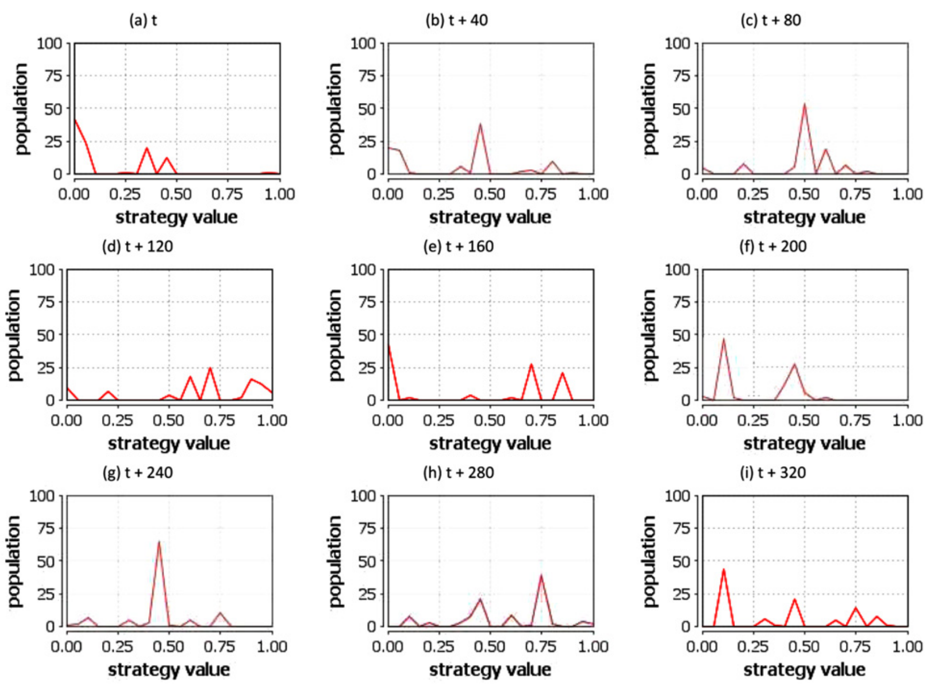

Fig. 5. The figure shows the population dynamics of APA with $n = 3$. The presented snapshots taken in the interval of 40 generations after running already for $t = 10^6$ generations show how individuals with higher bids are invading again and again up to a certain point where low bidding individuals can take over the population and the cycle will start again.

three participants. In both cases, the snapshots were taken as follows: after running for 10^6 generations (when we are close to the stationary distribution), we took the snapshots in the interval of 40 generations. We observe that in the stochastic computer simulation with a finite population, the strategy distribution is not static. The evolutionary dynamics driven by the selection–mutation process show a cyclic behavior within stochastic fluctuations. The cyclic behavior is as follows: at some stage the population is dominated by low bidding strategists; then with mutation and selection the population starts shifting slowly towards higher bidding strategists. This is because if there are a lot of low bidding strategists, then higher bids have better payoff as they slightly outbid the lower bidders and therefore get all the benefits and incur nearly similar costs. But then at the point when the population is dominated by very high bidding strategists, low bidding strategists start having better payoff as the high bidders incur huge costs when they participate in the same auctions, and thus with mutation and selection the population shifts towards low bids; we have the cyclic behavior as illustrated in Figs. 5 and 6. But very interestingly if we average over the strategy

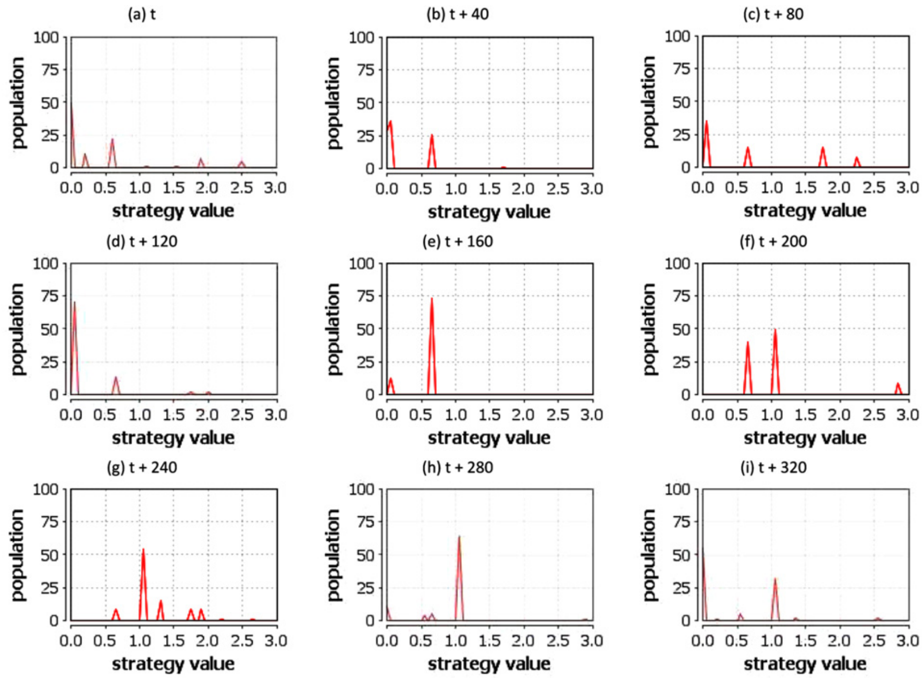

Fig. 6. We show the population dynamics of SAPA for $n = 3$. The presented snapshots taken in the interval of 40 generations after running already for $t = 10^6$ generations. As in Fig. 5, the figure shows the cycle of bidding from low bids to high bids and then coming back. We show only bids in the interval [0, 3] as higher bidding individuals are very rarely present in the population.

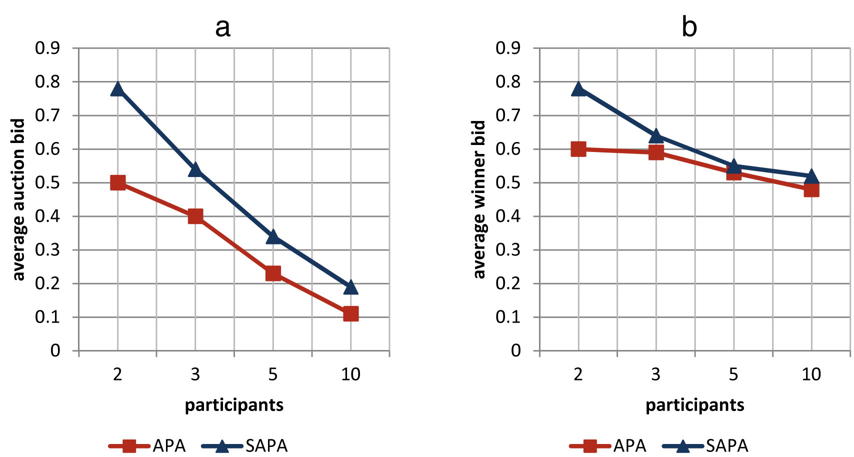

Fig. 7. Fig. 7(a) shows the average bid in APA and SAPA. The red color represents APA and the blue color represents SAPA. The average bids of the participating individuals are decreasing with an increasing number of participants. Furthermore, we observe that the difference of the bids in APA and SAPA decreases when n increases what we already have shown with the ESS curves. Fig. 7(b) shows the average bid of the winners of the auction for APA and SAPA. The winner bids are again higher in SAPA, especially when $n = 2$. The difference to APA is already small when $n = 3$, and almost identical for $n = 5$ and $n = 10$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

distribution, we observe that they converge to the ESS distribution. In other words, in the stochastic finite population though there is no static equilibrium, in the long run the average converges to the theoretical ESS distribution. Thus our simulation results show excellent agreement with the analytical results (Figs. 3 and 4), and is a key finding of our paper.

The average bid graphs. The average bid of all players and the average bid of the winner in the auctions for $n = 2, 3, 5$, and 10 are shown in Fig. 7.

6. Conclusion

In this work, we have studied the evolutionary dynamics of all-pay auctions and second price all-pay auctions. SAPA generalize the well known "war of attrition". We present analytical results (i) for the case when there are only two strategies, and (ii) when there is a continuum of strategies. We simulated evolutionary dynamics for both types of auctions for a finite population size. For both APA and SAPA, we show that the average strategy distribution converges to the mixed ESS in our stochastic computer simulation. APA and SAPA are different though specially for small numbers of participants: for example, for $n = 2$, the mixed equilibrium distribution for APA is uniform and for SAPA it is an exponential decay distribution; the average bid for SAPA is twice that of APA. As n becomes larger, the mixed ESS distribution and the average bid of the SAPA approaches the results of APA. We also observe that as the number of participants for the auction increases, the average bid declines both for APA and SAPA. Moreover, we observe the very interesting phenomenon that as n goes to ∞, any strategy is *invadable* by any other strategy. The reason is as follows: consider a population where all players bid 0.5, and a player with any other strategy. If the other strategy has a value greater than 0.5, then the other strategy always wins and has a better payoff. If the other strategy has a lower value, then the strategy does not win, but since n is large, every other player rarely wins, but always pays a greater cost, and hence again the new strategy has a better payoff.

Acknowledgments

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Appendix A. Details of Section 3.1

Proof of Lemma 1. We prove the two expected payoffs as follows.

(a) The proof of payoff expressed in Eq. (1) is as follows:

$$\begin{aligned} p_{\text{APA}}(s_1) &= x^{n-1} \cdot \left(\frac{V}{n} - s_1\right) + (1 - x^{n-1}) \cdot (-s_1) \\ &= \frac{x^{n-1} \cdot V}{n} - s_1 \cdot x^{n-1} - s_1 + s_1 \cdot x^{n-1} \\ &= \frac{x^{n-1} \cdot V}{n} - s_1; \end{aligned}$$

i.e., with probability x^{n-1} all individuals are selected as s_1 and then the probability of winning is $\frac{1}{n}$ (expected payoff then is $\frac{V}{n} - s_1$), and otherwise there is no probability to win and the payoff is $-s_1$.

(b) We will now prove the payoff of s_2 strategists given in Eq. (2). The payoff for an s_2 individual is

$$p_{\text{APA}}(s_2) = \sum_{i=0}^{n-1} \binom{n-1}{i} \cdot x^{n-1-i} \cdot (1-x)^i \cdot \left(\frac{V}{i+1} - s_2\right)$$

i.e., we calculate the probability of selected i of other s_2 individual and then the probability of winning is $\frac{1}{i+1}$ and expected payoff is $\frac{V}{i+1} - s_2$, and this is done for all $0 \le i \le n-1$. We now simplify the above expression.

$$\begin{aligned} p_{\text{APA}}(s_2) &= \sum_{i=0}^{n-1} \binom{n-1}{i} \cdot x^{n-1-i} \cdot (1-x)^i \cdot \left(\frac{1}{i+1} - s_2\right) \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} \cdot x^{n-1-i} \cdot (1-x)^i \cdot \left(\frac{V}{i+1}\right) - s_2 \\ &\quad \cdot \sum_{i=0}^{n-1} \binom{n-1}{i} \cdot x^{n-1-i} \cdot (1-x)^i \\ &= \frac{V \cdot (1-x^n)}{n \cdot (1-x)} - s_2 \\ &\quad \cdot \underbrace{\sum_{i=0}^{n-1} \binom{n-1}{i} \cdot x^{n-1-i} \cdot (1-x)^i}_{\text{usingbinomialtheorem}: (x+(1-x))^{n-1}=1} \\ &= \frac{V \cdot (1-x^n)}{n \cdot (1-x)} - s_2 \end{aligned}$$

where we used the fact that

$$\begin{aligned} &\sum_{i=0}^{n-1} \binom{n-1}{i} \cdot x^{n-1-i} \cdot (1-x)^i \cdot \left(\frac{1}{i+1}\right) \\ &= \sum_{i=0}^{n-1} \frac{(n-1)!}{i! \cdot (n-1-i)! \cdot (i+1)} \cdot x^{n-1-i} \cdot (1-x)^i \\ &= \frac{1}{n} \cdot \sum_{i=0}^{n-1} \frac{n \cdot (n-1)!}{(i+1) \cdot i! \cdot (n-(i+1))!} \cdot x^{n-(i+1)} \cdot (1-x)^i \\ &= \frac{1}{n \cdot (1-x)} \cdot \sum_{i=0}^{n-1} \binom{n}{i+1} \cdot x^{n-(i+1)} \cdot (1-x)^{i+1} \\ &= \frac{1}{n \cdot (1-x)} \cdot \sum_{j=1}^{n} \binom{n}{j} \cdot x^{n-j} \cdot (1-x)^j \\ &\quad (\text{rewriting } i+1 \text{ as } j) \\ &= \frac{1}{n \cdot (1-x)} \cdot (1-x^n) = \frac{1-x^n}{n \cdot (1-x)}; \end{aligned}$$

where in the last equality we used that $\sum_{k=0}^{n} \binom{n}{k} \cdot x^{n-k} \cdot (1-x)^k = 1$.

Thus the result of Lemma 1 follows. □

Appendix B. Details of Section 3.2

Proof of Theorem 2. Our goal is to obtain $p(x)$ such that $\frac{\delta E(s,I)}{\delta s} = 0$. Let $f(s) = \int_0^s p(x)dx$. Then we have

$$E(s, I) = V \cdot f(s)^{n-1} - s;$$

i.e., the strategy s wins value V with probability $f(s)^{n-1}$, where all other $n-1$ players bid less than s, and always pays the bid s. Hence we have

$$\frac{\delta E(s, I)}{\delta s} = V \cdot (n-1) \cdot f(s)^{n-2} \cdot f'(s) - 1.$$

Since we want $\frac{\delta E(s,I)}{\delta s} = 0$ we must have

$$(n-1) \cdot f(s)^{n-2} \cdot f'(s) = \frac{1}{V}.$$

Integrating both sides, we obtain that

$$\int (n-1) \cdot f(s)^{n-2} \cdot f'(s)ds = \int \frac{1}{V}ds.$$

Hence, we have

$$(n-1) \cdot \frac{f(s)^{n-1}}{(n-1)} = \frac{s}{V}.$$

Thus $f(s) = \left(\frac{s}{V}\right)^{\frac{1}{n-1}}$, and hence $f'(s) = \left(\frac{1}{V}\right)^{\frac{1}{n-1}} \cdot \frac{1}{n-1} \cdot s^{\frac{2-n}{n-1}}$. Thus we obtain the following solution for $p(x)$:

$$p(x) = \left(\frac{1}{V}\right)^{\frac{1}{n-1}} \cdot \frac{1}{n-1} \cdot x^{\frac{2-n}{n-1}}.$$

It follows that the above density function $p(x)$ is a mixed equilibrium.

We now study when the mixed equilibrium is an ESS. Let I be a mixed strategy and J be any other pure strategy. We will use the following notation: we denote by $E(X, (Y^i, Z^j))$ the expected payoff of the strategy X, against i players with strategy Y and j players with strategy Z, where $X, Y, Z \in \{I, J\}$, and $i + j = n - 1$ since there are n participants for the auction. A mixed strategy I is an ESS iff one of the following two conditions holds for all strategies J different from I:

(a) $E(I, (I^{n-1}, J^0)) > E(J, (I^{n-1}, J^0))$; or
(b) $E(I, (I^{n-1}, J^0)) = E(J, (I^{n-1}, J^0))$ and $E(I, (I^{n-2}, J^1)) > E(J, (I^{n-2}, J^1))$.

The first condition is similar to the definition of a *strict Nash equilibrium*. The second condition states that even when the strategy J is neutral against strategy I, I has an advantage when playing against J and therefore, there is no long-term incentive to switch to strategy J; for the conditions, see (Maynard Smith and Price, 1973; Maynard Smith, 1974, 1982; Haigh and Cannings, 1989).

We have shown that the mixed equilibrium I is given by the following distribution

$$p(x) = \left(\frac{1}{V}\right)^{\frac{1}{n-1}} \cdot \frac{1}{n-1} \cdot x^{\frac{2-n}{n-1}}.$$

We now calculate the following four quantities for a pure strategy J with value s.

(a) $E(I, (I^{n-1}, J^0))$: we have

$$E(I, (I^{n-1}, J^0)) = \frac{1}{n} \cdot V - \int_0^V x \cdot p(x)dx = \frac{1}{n} \cdot V - \frac{1}{n} \cdot V = 0.$$

(b) $E(J, (I^{n-1}, J^0))$: we have

$$E(J, (I^{n-1}, J^0)) = V \cdot \left(\int_0^s p(x)dx\right)^{n-1} - s = V \cdot \left[\left(\frac{1}{V}\right)^{\frac{1}{n-1}} \cdot s^{\frac{1}{n-1}}\right]^{n-1} - s = s - s = 0.$$

(c) $E(I, (I^{n-2}, J^1))$: we have

$$E(I, (I^{n-2}, J^1)) = V \cdot \frac{1}{n-1} \cdot \underbrace{\left[1 - \left(\int_0^s p(x)dx\right)^{n-1}\right]}_{\text{prob. of } J \text{ does not win}} - \int_0^V x \cdot p(x)dx = \frac{V}{n-1} \cdot (1 - \frac{s}{V}) - \frac{1}{n} \cdot V = \frac{V - s \cdot n}{n \cdot (n-1)}.$$

(d) $E(J, (I^{n-2}, J^1))$: we have

$$E(J, (I^{n-2}, J^1)) = \frac{V}{2} \cdot \left(\int_0^s p(x)dx\right)^{n-2} - s = \frac{V^{\frac{1}{n-1}} \cdot s^{\frac{n-2}{n-1}}}{2} - s.$$

It follows from the above calculation that we have $E(I, (I^{n-1}, J^0)) = E(J, (I^{n-1}, J^0))$, i.e., the first condition of ESS is not met. Hence, we focus whether the second ESS condition is met or not. We consider

$$f(s) = E(I, (I^{n-2}, J^1)) - E(J, (I^{n-2}, J^1)) = \frac{V - s \cdot n}{n \cdot (n-1)} - \frac{V^{\frac{1}{n-1}} \cdot s^{\frac{n-2}{n-1}}}{2} + s.$$

For the second ESS condition to be true, we need to check whether $f(s)$ is always positive. First we observe that for $n = 2$, we have $f(s) = 0$. It follows that for $n = 2$, there is no ESS. However, we show that for all $n > 2$, the second ESS condition is satisfied (i.e., $f(s)$ is always positive). To show this, we take the first and the second derivative of $f(s)$.

$$f'(s) = 1 - \frac{1}{n-1} - \frac{V^{\frac{1}{n-1}} \cdot (n-2)}{2 \cdot (n-1)} \cdot s^{-\frac{1}{n-1}}$$

$$f''(s) = \frac{V^{\frac{1}{n-1}} \cdot (n-2)}{2 \cdot (n-1)^2} \cdot s^{-\frac{n}{n-1}}.$$

Observe that for $n > 2$ and s positive, the second derivative $f''(s)$ is positive. Hence, if we consider $f(s)$ at s^* where $f'(s^*) = 0$, then we obtain the minima of $f(s)$. We take the first derivative $f'(s)$ and set it equal to zero to obtain $s^* = \frac{V}{2^{n-1}}$. We have

$$f(s^*) = \frac{V - V \cdot 2^{1-n} \cdot n}{n \cdot (n-1)} - \frac{V^{\frac{1}{n-1}} \cdot V^{\frac{n-2}{n-1}} \cdot 2^{1-n \cdot \frac{n-2}{n-1}}}{2} + V \cdot 2^{1-n} = \frac{V \cdot \left(1 - \frac{n}{2^{n-1}}\right)}{n \cdot (n-1)}.$$

We observe that $f(s^*)$ is positive for all $n > 2$, and hence it follows that $f(s)$ is always positive for all $n > 2$. □

Lifetime investment reduction. We show how lifetime investments of individuals (like in Example 1) can be reduced to our model of APA with bids in each competition. Let k be the expected number of times that an individual participates in auctions throughout its whole life, and let the lifetime investment be of cost c. It is important to note that in our model an individual cannot influence the number of participations in such competitions. Hence, we can split the lifetime investment of the individual over all competitions and then the bid of this individual in each auction is c/k (i.e., the individual incurs costs of c/k in each competition).

Appendix C. Details of Section 4.1

Proof of Lemma 2. For s_1 strategists, the payoff of SAPA coincide with the payoff for APA. Below we prove Eq. (10). Since the only difference to APA is if all other $n-1$ participants are playing s_1, the payoffs are

$$\begin{aligned} p_{2^{\text{nd}}}(s_2) &= \sum_{i=1}^{n-1} \binom{n-1}{i} \cdot x^{n-1-i} \cdot (1-x)^i \cdot \left(\frac{V}{i+1} - s_2\right) + x^{n-1} \cdot (1 - s_1) \\ &= \frac{V \cdot (1-x^n)}{n \cdot (1-x)} - x^{n-1} - s_2 \cdot \underbrace{\sum_{i=1}^{n-1} \binom{n-1}{i} \cdot x^{n-1-i} \cdot (1-x)^i}_{(x+(1-x))^{n-1} - x^{n-1} = 1 - x^{n-1}} + x^{n-1} \cdot (1 - s_1) \\ &= \frac{V \cdot (1-x^n)}{n \cdot (1-x)} - x^{n-1} - s_2 \cdot (1 - x^{n-1}) + x^{n-1} \cdot (1 - s_1) \\ &= \frac{V \cdot (1-x^n)}{n \cdot (1-x)} - s_2 + x^{n-1} \cdot (s_2 - s_1) \end{aligned}$$

where we used the same fact as before that

$$\begin{aligned} &\sum_{i=1}^{n-1} \binom{n-1}{i} \cdot x^{n-1-i} \cdot (1-x)^i \cdot \left(\frac{1}{i+1}\right) \\ &= \sum_{i=1}^{n-1} \frac{(n-1)!}{i! \cdot (n-1-i)! \cdot (i+1)} \cdot x^{n-1-i} \cdot (1-x)^i \\ &= \frac{1}{n \cdot (1-x)} \cdot \sum_{i=1}^{n-1} \binom{n}{i+1} \cdot x^{n-(i+1)} \cdot (1-x)^{i+1} \\ &= \frac{1}{n \cdot (1-x)} \cdot \sum_{j=2}^{n} \binom{n}{j} \cdot x^{n-j} \cdot (1-x)^j \cdot \frac{1}{n \cdot (1-x)} \cdot \left[1 - (x^n + n \cdot (x^{n-1} - x^n))\right] \\ &= \frac{1 - (x^n + n \cdot x^{n-1} \cdot (1-x))}{n \cdot (1-x)} = \frac{1 - x^n}{n \cdot (1-x)} - x^{n-1}. \quad \square \end{aligned}$$

The desired result follows. □

Proof of Eq. (11).

$$\frac{V \cdot x^{n-1}}{n} - s < \frac{V \cdot (1-x^n)}{n \cdot (1-x)} - s' + x^{n-1} \cdot (s' - s)$$
$$s' - s - x^{n-1} \cdot (s'-s) < V \cdot \left(\frac{1-x^n}{n \cdot (1-x)} - \frac{x^{n-1}}{n} \right)$$
$$s' \cdot (1-x^{n-1}) - s \cdot (1-x^{n-1}) < V \cdot \left(\frac{1-x^{n-1}}{n \cdot (1-x)} \right)$$
$$s' - s < V \cdot \left(\frac{1-x^{n-1}}{n \cdot (1-x) \cdot (1-x^{n-1})} \right)$$
$$s' - s < \frac{V}{n \cdot (1-x)}. \quad \square$$

Appendix D. Details of Section 4.2

In this section, we show how to derive the mixed equilibrium for SAPA.

Proof of Theorem 4. We present the proof of Theorem 4. We present a detailed derivation of the mixed equilibrium and the fact that the equilibrium is an ESS for all $n \geq 2$ follows from the results of (Haigh and Cannings, 1989). We compute the price to pay for the winner. First, we calculate the probability of paying x as the 2nd highest bid.

- Let $\alpha = p(x)\delta x$ be the probability of a selection in $[x, x+\delta x]$. As δx is very small ($\delta x \to 0$), we will ignore α^i for $i > 1$.
- Let $\beta(x) = \int_0^x p(x)dx$ be the probability of selecting less than x.
- Hence, the probability to pay x is
$$\sum_{i=1}^{n-1} \binom{n-1}{i} \cdot \alpha^i \cdot \beta(x)^{n-1-i}.$$
- Since we ignore α^i for $i > 1$, the probability to pay x is
$$(n-1) \cdot \alpha \cdot \beta(x)^{n-2}.$$

The total payment for this case is then

$$(n-1) \cdot \int_0^s x \cdot p(x) \cdot \left(\int_0^x p(x)dx \right)^{n-2} dx.$$

Hence we obtain the expected payoff $E(s, I)$ as follows:

$$E(s,I) = V \cdot \left(\int_0^s p(x)dx \right)^{n-1} - s \cdot \left[1 - \left(\int_0^s p(x)dx \right)^{n-1} \right] - (n-1) \cdot \int_0^s x \cdot p(x) \cdot \left(\int_0^x p(x)dx \right)^{n-2} dx.$$

Note that with $n = 2$, the above $E(s, I)$ reduce to

$$V \cdot \int_0^s p(x)dx - s \cdot \left(1 - \int_0^s p(x)dx \right) - \int_0^s x \cdot p(x)dx = \int_0^s (V - x) \cdot p(x)dx - s \cdot \int_s^\infty p(x)dx$$

which is exactly as given by Maynard Smith for war of attrition.

Solution of differential equation for $p(x)$. We will find $p(x)$ such that $\frac{\delta E(s,I)}{\delta s} = 0$. We first present a few notations

$$h(x) = \int p(x)dx; \qquad g(x) = p(x) \cdot \left(\int_o^x p(x)dx \right)^{n-2};$$
$$h_1(x) = \int g(x)dx; \qquad h_2(x) = \int h_1(x)dx.$$

Let

$$\ell(x) = (n-1) \cdot (x \cdot h_1(x) - h_2(x)) = (n-1) \int x \cdot g(x)dx.$$

Then we have

$$\frac{\delta}{\delta s}(\ell(s) - \ell(0)) = (n-1) \cdot (h_1(s) + s \cdot h_1'(s) - h_2'(s)) = (n-1) \cdot s \cdot g(s);$$

in the above equation, we have used that $h_1'(s) = g(s)$ and $h_2'(s) = h_1(s)$. We have the following

$$\frac{\delta}{\delta s} \left(-s \cdot \left(1 - \left(\int_0^s p(x)dx \right)^{n-1} \right) \right) = -(1 - (h(s) - h(0))^{n-1}) + s \cdot (n-1) \cdot (h(s) - h(0))^{n-2} \cdot h'(s),$$

and since $h'(s) = p(s)$ we obtain that

$$\frac{\delta}{\delta s} \left(-s \cdot \left(1 - \left(\int_0^s p(x)dx \right)^{n-1} \right) \right) = -(1 - (h(s) - h(0))^{n-1}) + s \cdot (n-1) \cdot g(s).$$

Thus we obtain that

$$\frac{\delta E(s,I)}{\delta s} = V \cdot (h(s) - h(0))^{n-2} \cdot h'(s) - 1 + (h(s) - h(0))^{n-1} + s \cdot (n-1) \cdot g(s) - s \cdot (n-1) \cdot g(s).$$

Thus we need to solve the differential equation that

$$V \cdot (h(s) - h(0))^{n-2} \cdot h'(s) - 1 + (h(s) - h(0))^{n-1} = 0.$$

To solve the above differential equation let $y(s) = h(s) - h(0)$, and thus we must have $y(0) = 0$. The above differential equation is re-written as

$$V \cdot (n-1) \cdot y^{n-2}dy = (1 - y^{n-1})ds \Rightarrow V \cdot \frac{(n-1) \cdot y^{n-2}}{1 - y^{n-1}}dy = ds.$$

In the above equation, integrating both sides we obtain that

$$-V \cdot \ln(1 - y^{n-1}) = s + C,$$

where C is a constant. Thus $y(s) = (1 - \exp(-\frac{s}{V}))^{\frac{1}{n-1}}$, and since $y(s) = 0$ we have $C = 0$. Hence we obtain the following: $h(s) = (1 - \exp(-\frac{s}{V}))^{\frac{1}{n-1}} - 1$, and $h(0) = -1$ and then we have

$$p(s) = h'(s) = \frac{1}{(n-1) \cdot V} \cdot \left(1 - \exp\left(-\frac{s}{V} \right) \right)^{\frac{2-n}{n-1}} \cdot \exp\left(-\frac{s}{V} \right).$$

This completes the proof of Theorem 4. $\square$

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