

## Markov Chains or Markov Processes

A finite **Markov chain** or **Markov process** describes a closed system in which objects move from one state to another by a pre-determined probability. In general, the system can be written as:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$$

where  $\mathbf{x}_k$  is the vector containing the number of objects at different states at time  $k$  and  $\mathbf{A}$  is the *transition matrix* containing the probability of objects moving from one state to another. During any given time period, each object is in only one state.

For the system to be a Markov chain, the transition matrix  $\mathbf{A}$  must satisfy the following properties:

1. all elements of  $\mathbf{A}$  are positive:

Element  $a_{ij}$  represents the probability of an object in state  $j$  switching to state  $i$ . Since probability is always positive, each element of  $\mathbf{A}$  must be positive.

2. all columns of  $\mathbf{A}$  must add up to 1.

Because Markov chains describe closed systems (no creation or destruction of “object”), the total number of objects must be conserved throughout the time steps.

These two conditions together lead to the property that the absolute value of the eigenvalues of  $\mathbf{A}$  are always less than or equal to 1 with at least one eigenvalue = 1.

**Example:** Consider the example of population of California and the rest of the United States we discussed in class. We had:

$$\begin{pmatrix} y_{k+1} \\ z_{k+1} \end{pmatrix} = \begin{pmatrix} 9/10 & 2/10 \\ 1/10 & 8/10 \end{pmatrix} \begin{pmatrix} y_k \\ z_k \end{pmatrix}$$

where  $y_k$  was the non-California population at time  $k$  and  $z_k$  was the population of California at time  $k$ . Here the “object” is the population and there are two “states” (not geographical states), one is California and another is outside California.

$a_{12} = 2/10$  is the probability of “object” in state 1 (population outside California) to switch to state 2 (California).

The columns add up to 1  $\Rightarrow$  we have a closed system (no increase or decrease in total population).

**Example:** Every day people in Boston travel to work either by car or MBTA. 70% of the people who drove to work the previous day drive to work the next day, whereas 30% of the people who drove to work got sick of the traffic and decided to take the “T” the next day. On the other hand, 80% of the people who took the “T” the previous day continue to take the “T” the next day whereas 20% of the people get sick of the congestion on the trains and decide to drive to work the next day. The same pattern is observed everyday. Find the distribution of travellers after  $k$  days and after a very long time.

Let  $c_0$  and  $t_0$  be the number of car travellers and train travellers respectively on day 0. Then after 1 day the distribution is:

$$\begin{aligned} c_1 &= 7/10c_0 + 2/10t_0 \\ t_1 &= 3/10c_0 + 8/10t_0 \end{aligned} \Rightarrow \begin{pmatrix} c_1 \\ t_1 \end{pmatrix} = \begin{pmatrix} 7/10 & 2/10 \\ 3/10 & 8/10 \end{pmatrix} \begin{pmatrix} c_0 \\ t_0 \end{pmatrix} \text{ i.e., } \mathbf{A} = \begin{pmatrix} 7/10 & 2/10 \\ 3/10 & 8/10 \end{pmatrix}$$

after k days we get:  $\begin{pmatrix} c_k \\ t_k \end{pmatrix} = \begin{pmatrix} 7/10 & 2/10 \\ 3/10 & 8/10 \end{pmatrix}^k \begin{pmatrix} c_0 \\ t_0 \end{pmatrix}$

now, we can calculate  $\mathbf{A}^k$  using diagonalization:  $\mathbf{A}^k = \mathbf{X}\mathbf{\Lambda}^k\mathbf{X}^{-1}$

Find eigenvalues to make  $\mathbf{\Lambda}$ :  $\begin{vmatrix} 7/10 - \lambda & 2/10 \\ 3/10 & 8/10 - \lambda \end{vmatrix} = 0 \Rightarrow (7 - 10\lambda)(8 - 10\lambda) - 6 = 0$   
 $(\lambda - 1)(2\lambda - 1) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 1/2$

Find eigenvectors to make  $\mathbf{X}$

$$\lambda_1 = 1: \begin{pmatrix} -3/10 & 2/10 \\ 3/10 & -2/10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{x}_1 = t \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\lambda_2 = 1/2: \begin{pmatrix} 2/10 & 2/10 \\ 3/10 & 3/10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{x}_2 = t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \text{ and } \mathbf{X} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \Rightarrow \mathbf{X}^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}^k &= \mathbf{X}\mathbf{\Lambda}^k\mathbf{X}^{-1} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (1/2)^k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 2 + 3(2^{-k}) & 2 - 2(2^{-k}) \\ 3 - 3(2^{-k}) & 3 + 2(2^{-k}) \end{pmatrix} \end{aligned}$$

There are 2 checks you can do. first check for  $k = 0$  since we expect  $\mathbf{A}^0 = \mathbf{I}$ . Secondly check that  $k = 1$  gives back the matrix  $\mathbf{A}$

So after k days:

$$\begin{pmatrix} c_k \\ t_k \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 + 3(2^{-k}) & 2 - 2(2^{-k}) \\ 3 - 3(2^{-k}) & 3 + 2(2^{-k}) \end{pmatrix} \begin{pmatrix} c_0 \\ t_0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} [2 + 3(2^{-k})]c_0 + [2 - 2(2^{-k})]t_0 \\ [3 - 3(2^{-k})]c_0 + [3 + 2(2^{-k})]t_0 \end{pmatrix}$$

$$\Rightarrow \{[2 + 3(2^{-k})]c_0 + [2 - 2(2^{-k})]t_0\}/5 \text{ people will travel to work by car after k days}$$

$$\Rightarrow \{[3 - 3(2^{-k})]c_0 + [3 + 2(2^{-k})]t_0\}/5 \text{ people will travel to work by train after k days}$$

after very very many days, i.e.,  $k \rightarrow \infty$

$$\mathbf{A}^\infty = \frac{1}{5} \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

$$\begin{pmatrix} c_\infty \\ t_\infty \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} c_0 \\ t_0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2c_0 + 2t_0 \\ 3c_0 + 3t_0 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix} (c_0 + t_0)$$

Note that  $c_0 + t_0$  represents the total number of travellers.

$\Rightarrow$  after very very many days, 40% of the total travellers will be travelling by car and 60% of the total travellers will be travelling by MBTA trains.