1 Random Walks

Exercise. *This weekend, you decide to go to a casino and gamble. You start with j dollars, and you decide that if you ever have $k \geq j$ dollars, you will take your winnings and go home. Assuming that at each step you either win \$1 or lose \$1 (with equal probability), what is the probability that you end up with \$$k$? What's the probability that instead you lose all your money?*

Let's set this up as follows: Let P_i be the probability of winning if you currently have \$i. Then $P_0 = 0, P_k = 1, P_i = \frac{1}{2}P_{i-1} + \frac{1}{2}P_{i+1}$. Trial and error gives us $P_j = {}^{j}/_{k}$ – try it for yourself. Then the probability of losing all your money instead is the probability of not winning, i.e. $1 - {}^{j}/_{k}$.

Exercise. *Now suppose that there are $n + 1$ people in a circle numbered $0, 1, \ldots, n$. Person 0 starts with a bag of candy. At each step, the person with the bag of candy passes it either left or right with equal probability. The* last *person to receive the bag wins (and gets to keep all the candy). What is the probability that person i wins?*

We're given that $Pr(\text{Person 0 wins}) = 0$.

What's $Pr(\text{Person } n \text{ wins})$? In order for n to win, the candy must start from 0 and get all the way around to $n - 1$ before ever getting to n (in order for everyone else to have gotten the candy before n ever does).

But this is equivalent to the casino problem we just did! Starting at 0 and getting to $n - 1$ before ever getting to n, with equal probability of going left or right, is analogous to starting at \$1 in the casino game and getting to \$$k$ before ever getting to 0. Then $Pr(\text{Person } n \text{ wins}) = \frac{1}{n}$. By symmetry, $Pr(\text{Person 1 wins})$ is also $\frac{1}{n}$.

What about everyone else? Well, here's the trick: For any person i to win, the people on either side of i must have gotten the bag before i ever does. Then there are two cases – either $i - 1$ gets the bag before $i+1$ does, or $i+1$ gets the bag before $i-1$ does. So the probability that i wins is the probability of getting to the first case times the probability of i winning in the first case, plus the probability of getting to the second case times the probability of i winning in the second case. We can write this as follows:

$$Pr(i \text{ wins}) = \begin{array}{ll} & Pr(i+1 \text{ gets bag before } i-1) \times Pr(i \text{ wins} \mid i+1 \text{ gets bag before } i-1) \\ + & Pr(i-1 \text{ gets bag before } i+1) \times Pr(i \text{ wins} \mid i-1 \text{ gets bag before } i+1) \end{array}$$

(If you don't remember, the things with the $|$'s are **conditional probabilities**: what's the probability that i wins *given that* $i - 1$ gets the bag before $i + 1$, and what's the probability that i wins *given that* $i + 1$ gets the bag before $i - 1$?)

All right, now let's try and compute these conditional probabilities. First, let's compute the probability that i wins, given that $i - 1$ gets the bag before $i + 1$ does. Here comes another trick: Consider the *first* time that person $i - 1$ gets the bag, in those circumstances when person $i + 1$ has not gotten the bag yet. (At that moment, person i cannot have gotten the bag yet, because up until this point, neither person on either side of i has had the bag!)

Then starting at that point, what do we have? We need for the bag to go all the way around from $i-1, i-2, i-3, \ldots$ to $\ldots i+3, i+2, i+1$ without ever going the wrong way, from $i - 1$ to i – it's the casino problem again! Going from $i - 1$ to $i + 1$ without ever going to i is again analogous to starting at \$1 in the casino game and getting to \$$n$ without ever hitting \$0. Therefore, $Pr(i \text{ wins} \mid i+1 \text{ gets bag before } i-1) = \frac{1}{n}$.

And by symmetry, $Pr(i-1 \text{ gets bag before } i+1) \times Pr(i \text{ wins} \mid i-1 \text{ gets bag before } i+1) = \frac{1}{n}$ too. Then we have

$$Pr(i \text{ wins}) = \begin{array}{ll} & Pr(i+1 \text{ gets bag before } i-1) \times {}^{1}/_{n} \\ + & Pr(i-1 \text{ gets bag before } i+1) \times {}^{1}/_{n} \end{array}$$

Rearranging, we have:

$Pr(i \text{ wins}) = {}^{1}/_{n} \times [Pr(i+1 \text{ gets bag before } i-1) + Pr(i-1 \text{ gets bag before } i+1)]$

Now, what do we do about $Pr(i+1 \text{ gets bag before } i-1)$ and $Pr(i-1 \text{ gets bag before } i+1)$? Well, it's not entirely easy to calculate each, but we know that one or the other must happen (either $i+1$ gets the bag before $i-1$, or $i-1$ gets the bag before $i+1$), so those probabilities sum to 1!

So in the end, we find that the probability that person i wins is $\frac{1}{n}$ for $i \neq 0$.