1 Expected Value

Suppose that a random variable X doubles at each timestep with probability $1/2$ and halves with probability $1/2$. Show that $E[\log X_{t+1}] = E[\log X_t]$.

$$\begin{aligned}
E[\log(X_t)] &= 1/2E[\log(X_t/2)] + 1/2E[\log(2X_t)] \\
&= 1/2(E[\log X_t - \log 2] + E[\log 2 + \log X_t]) \\
&= 1/2(E[\log X_t - \log 2 + \log 2 + \log X_t]) \\
&= E[1/2(\log X_t - \log 2 + \log 2 + \log X_t)] \\
&= E[\log X_t]
\end{aligned}$$

Suppose that $X_0 = 1$, and when the game ends, $X = 8$ or $X = 1/8$. Find the probability p_0 that $X = 8$. $E[\log X_t]$ remains constant at 0, since $\log X_0 = 0$. So

$$\begin{aligned}
0 &= E[\log(X_{end})] \\
&= p_0 \log 8 + (1 - p_0) \log 1/8 \\
&= 3p_0 - 3(1 - p_0) \\
p_0 &= 1/2
\end{aligned}$$