SP2 Proof

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Note: We use "SP2" to refer to the algorithm for "Single-Source Shortest Paths: General Lengths" given on page 5 of the Lecture 4 notes.

First, we define $SP(x)$ to be the shortest path from node s to node x with fewest edges. (i.e. if there is more than one shortest path, choose the shortest path with fewest edges.) Let $\circ$ denote concatenation of paths.

Lemma: After iterating i times, SP2 finds all shortest paths of i edges or fewer.
Proof: By induction on i.
Base Case: $i = 1$. Suppose $SP(x)$ has one edge. Then $SP(x) = (s, x)$. After one iteration, SP2 will have updated (s, x), and so the *dists* array will have *length*(s, x) as the distance from s to x, and the *prev* array will have *prev*$(x) = s$. In other words, SP2 will have found $SP(s, x)$ after one iteration.
Induction Hypothesis: After iteration $i - 1$ times, SP2 finds all shortest paths of $i - 1$ edges or fewer.
Reduction Step: Suppose $SP(x)$ has j edges, where $j \leq i$. Then for some node k, $SP(x) = SP(k) \circ (k, x)$ (i.e. (k, x) is the last edge of $SP(x)$). Since $SP(x)$ has at most i edges, $SP(k)$ must have at most $i - 1$ edges, so after iterating $i - 1$ times, we have found $SP(k)$ (by the induction hypothesis). Then at the i^{th} iteration, we will update (k, x), and since $SP(k) \circ (k, x)$ is the shortest path to x, the value of *dists*(x) will be less than *dists*(k) + *length*(k, x) = *length*$(SP(k))$ + *length*(k, x), so we will update *dists* and *prev* so that the distance to x is *length*$(SP(s, k)$ + *length*(k, x) = *length*$(SP(s, x))$, and *prev*$(x) = k$. In other words, SP2 will have found $SP(s, x)$ after i iterations.

Proposition: SP2 finds all shortest paths on a graph with no negative cycles.
Proof: By the lemma, we need only to show that every node has a shortest path with no more than $V - 1$ edges. For sake of contradiction, suppose that $SP(x)$ has more than $V - 1$ edges. Then there must be some node y that is visited more than once, so there must be a cycle at y that is included in $SP(x)$. But we are assuming no negative cycles, so the cycle must have positive or 0 weight. If it has positive weight, then the length of $SP(x)$ can be reduced by removing the cycle, and if it has 0 weight, then it can be removed to give another shortest path with fewer edges, contradicting our definition of $SP(x)$ as the shortest path from s to x with fewest edges. So $SP(x)$ cannot have more than $V - 1$ edges.