1 Complexity Theory

Recall that $\mathbf{P}$ is the class of yes-no problems that can be solved in polynomial time. $\mathbf{NP}$ is the class of yes-no problems where a yes can be justified in polynomial time. For some examples, the following graph algorithms are in $\mathbf{P}$.

- Finding the SCC’s.
- Topological sort.
- Finding a minimum spanning tree.
- Finding the shortest path between two vertices.

In addition to all of those problems, the following graph algorithms are in $\mathbf{NP}$.

- Whether a graph contains a clique of size at least $k$.
- Whether there exists a path that visits all the vertices in a graph exactly once.
- Whether a graph can be colored with at most $k$ colors such that no two connected vertices are the same color.

These examples all allow for efficient certificates, as you could name the clique, the path, or the coloring, and fully convince someone that the answer is yes. In addition to being in $\mathbf{NP}$, all of these problems are $\mathbf{NP}$-complete. Recall that for a problem to be $\mathbf{NP}$-complete, it needs to

(a) Be an element of $\mathbf{NP}$.

(b) Have all other problems in $\mathbf{NP}$ reduce to it.

Intuitively, this means that the $\mathbf{NP}$-complete problems are the hardest problems in $\mathbf{NP}$.

2 Reductions

Exercise 1. Reduce the following problem to max-flow:

Call two paths *edge-disjoint* if they have no edges in common. Given a graph $G = (V, E)$ and two nodes $s$ and $t$, find the maximum number of edge-disjoint paths from $s$ to $t$. 
Exercise 2. Show that sorting $n$ integers can be reduced to the convex hull problem, which is as follows:

Given $n$ points in the plane, find the convex polygon of smallest area that contains all of the points. The polygon should be specified by listing the coordinates of its vertices (in order).

3 NP-Complete Reductions

Exercise 3. Reduce 3-SAT to 3-coloring. Recall that 3-SAT asks whether there is a way to assign truth values to $x_1, x_2, \ldots, x_n$ such that

$$(y_1 \lor y_2 \lor y_3) \land (y_4 \lor y_5 \lor y_6) \land \cdots \land (y_{3m-2} \lor y_{3m-1} \lor y_{3m})$$

evaluates to true, where each $y_i$ is $x_j$ or $\neg x_j$ for some $j$. 3-coloring is the problem of checking whether a graph can have each of its vertices colored one of three colors such that no two connected vertices share the same color.