Chapter 1

3D Geometry IV

1.1 Affine Invariance

The class of linear transformations gets its name because it is the class of transformations that is invariant to linear combinations

$$\begin{aligned} \mathcal{L}(\vec{v}+\vec{u}) &= \mathcal{L}(\vec{v})+\mathcal{L}(\vec{u}) \\ \mathcal{L}(\alpha\vec{v}) &= \alpha\mathcal{L}(\vec{v}) \end{aligned}$$

This class of transformations was exactly the set of transformations achieved when placing a 3 by 3 matrix between a basis and a coordinate vector. When an affine 4 by 4 matrix is placed between a frame and a homogeneous coordinate vector, we call this an affine transformation applied to a point. The reason why we use this name is because such transformations are invariant to affine combinations

$$\begin{aligned} \mathcal{A}(\beta_1\tilde{q}_1+\beta_2\tilde{q}_2) = \beta_1\mathcal{A}(\tilde{q}_1)+\beta_2\mathcal{A}(\tilde{q}_2) \\ \text{where}\beta_1+\beta_2 = 1 \end{aligned}$$

1.2 Barycentric basis and coordinates

There is an alternative to frames and homogeneous coordinates to specify points in A^3. This alternative is very useful for doing things like describing the postion of a point relative to some tetrahedra.

In this alternative framework, a point is specified by starting with 4 (non coplanar) basis points called a barycentric basis. One then describes any point as some affine combinatipon of these basis points.

$$\tilde{q} = \begin{bmatrix} \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

$$\text{where} \sum_i \alpha_i = 1$$

The only requirement of the barycentric coordinates is that they sum to one. When using frames, a 0 last homogeneous coordinate was used to represent a vector. When using a barycentric basis, a vector is specified if the barycentric coordinates sum to 0.

Barycentric coordinates are useful in lower dimemsions as well. In A^1 (the affine line) a point is specified by 2 barycentric coordinates. The barycentric coordinates describe where the point is relative to two chosen points.

In A^2, the affine plane, a point is specified by 3 barycentric coordinations. These barycentric coordinates describe where the point is relative to three chosen points. If the barycentric coordinates are all positive, then the point is inside the triangle described by the barycentric basis points.

There is a useful geometric interpretation of barycentric coordinates in A^2. Take a point inside a triangle. Draw segements from the point to the three vertices of the triangle. This produces three smaller triangles t_1, t_2, t_3. The the area of t_1 divided by the area of the original triangle is the first barycentric coodinate of the point. The other barycentric coordinates can be computed similarly.

Affine transformations can be expressed by inserting a matrix between a barycentric basis and barycentric coordinats.

$$\begin{aligned} & \begin{bmatrix} \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \\ \Rightarrow \quad & \begin{bmatrix} \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_4 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \\ = \quad & \begin{bmatrix} \tilde{p'}_1 & \tilde{p'}_2 & \tilde{p'}_3 & \tilde{p'}_4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \end{aligned}$$

For this to descibe a valid affine transformation, the sum of each of the four columns of the matrix must sum to 1.

1.3 Formal approach

The mathematicly inclined reader may note that we derived our notion of affine spaces in a somewhat informal way. Discussin in order, homogeneous coordinates, affine matrices, frames, and finally looking at affine invariance and barycentric bases.

A more formal approach would go in almost the opposite direction. One first defines an affine space as some set of elements that have a subtraction operation that returns an element in a vector space. This subtraction operation has to satisfy certain rules (just like vector addition need to satisfy some rules for linear spaces). One can use the subtraction operation to define the operation of affine combination of points. One then can define barycentric bases and coordinates. One then defines an affine transformation as a transformation that is affine invariant. One shows that this transformation class is exactly the class described by matices with columns summing to one. Finally one defines frames and homogeoneous coordinates as an alternative to specifying affine points. Finally one can show that the affine transformations are exactly those expressed by affine matrices.

Aren't you glad we didn't do it that way?

1.4 Gouraud shading revisited

In chapter **??** we described the gooro shading algorithm and stated that when applied to triangles, it is invariant to rotation. This can be trivially shown using the notion of affine invariance. In particular we can show that the color of a pixel at some point inside a triangle is an affine combination of the colors at the vertices. We can show that the affine weights used are exactly the barycentric coordinates of the pixel point when using the barycentric basis for A^2 defined by the three triangle vertices. When one rotates a triangle, one is simply applying an affine transformation to the points in A^2. Thus the new pixel point will be obtained using the same barycentric coordinates but relative to the rotated triangle. Thus this point will be colored with the same color.

Here are the gory details. First define the triangle with three points $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$ A point on the left edge of the triangle is uniquely defined by some affine combination $\alpha_1\tilde{p}_1 + \alpha_2\tilde{p}_2$ α_2 is the fraction of the way that i go from $\tilde{p}_1$ to $\tilde{p}_2$. Similarly, a point on the right edge is uniquely defined some the affine combination $\beta_1\tilde{p}_1 + \beta_2\tilde{p}_3$. Given these 2 points, a pixel point on the horizontal is uniquely defined by the affine combination

$$\begin{aligned}
\tilde{q} &= \gamma_1(\alpha_1\tilde{p}_1 + \alpha_2\tilde{p}_2) + \gamma_2(\beta_1\tilde{p}_1 + \beta_2\tilde{p}_3) \\
&= (\gamma_1\alpha_1 + \gamma_2\beta_1)\tilde{p}_1 + (\gamma_1\alpha_2)\tilde{p}_2 + (\gamma_2\beta_2)\tilde{p}_3 \\
&= \delta_1\tilde{p}_1 + \delta_2\tilde{p}_2 + \delta_3\tilde{p}_3
\end{aligned}$$

We can easily show that the sum of the δ is one

$$\begin{aligned}
\delta_1 + \delta_2 + \delta_3 & \\
&= \gamma_1\alpha_1 + \gamma_2\beta_1 + \gamma_1\alpha_2 + \gamma_2\beta_2 \\
&= (\gamma_1\alpha_1 + \gamma_1\alpha_2) + (\gamma_2\beta_1 + \gamma_2\beta_2) \\
&= \gamma_1(\alpha_1 + \alpha_2) + \gamma_2(\beta_1 + \beta_2) \\
&= \gamma_1 + \gamma_2 \\
&= 1
\end{aligned}$$

and so the δ are the barycentric coordinates of the pixle point using the triangle as the barycentric basis.

During the gouraud shading algorithm, we compute the left edge color as $\alpha_1 C_1 + \alpha_2 C_2$ and the right edge color as $\beta_1 C_1 + \beta_2 C_2$ where the C_i are the colors of the triangle verticies. The color of a point on the span is uniquely defined by the affine combination

$$\begin{aligned}
C_o &= \gamma_1(\alpha_1 C_1 + \alpha_2 C_2) + \gamma_2(\beta_1 C_1 + \beta_2 C_3) \\
&= \delta_1 C_1 + \delta_2 C_2 + \delta_3 C_3
\end{aligned}$$

So the color of the pixel is the affine combination of the colors, using the same weights as the barycentric coordinates of $\tilde{q}$

Rotating the triangle is an application of an affine transformation $\mathcal{A}$ to the triangle vertices as well as the pixel point. Under this transformation the new triangle vertices are $\mathcal{A}(\tilde{p}_i) = \tilde{p}'_i$

Since affine transformations preserve affine combinations, then the new pixel point must be

$$\tilde{q}' = \mathcal{A}(\sum_i \delta_i \tilde{p}_i) = \sum_i \delta_i \mathcal{A}(\tilde{p}_i) = \sum_i \delta_i \tilde{p}'_i$$

Thus, the rotated pixel point $\tilde{q}'$ has the same barycentric coordinates with resepct to the rotated triangle, as $\tilde{q}$ had with repsect to the original triangle. When gouraud shading is applied, the pixel point must be given the same color.