Chapter 1

Three Dimensional Geometry II

1.1 Points and Frames

It is useful thought to think of points and vectors as as two different concepts. When we think of a point, we think of a fixed place in a geometric world. When we think of a vector, we think of the motion between two points in the world. We will use two different notations to distinguish points and vectors. A vector $\vec{v}$ will have an arrow on top, while a point $\tilde{p}$ will have a squiggle on top. We will say that the vectors live in the linear space R^3, while the points live in the affine space A^3

If we think of a vector as representing motion between two points, then the vector operations, addition and scalar multiplication, have obvious meaning. If I add two vectors, I am expressing the concatination of two motions. If i multiply a vector by a scalar, I am increasing or decreasing the motion by some factor. The zero vector is a special vector that represents no motion.

These operations don't really make much sense for points. What should it mean to add two points together, eg. what is harvard square plus central sqaure? What does it mean to multiply a point by a scalar? What is 7 times the north pole? Is there a zero point that acts differently than the others?

There is one operation on two points that does sort of make sence, subtraction. When i subtract one point from another, I might say that i should get the motion that it takes to get from the second point to the first one,

$$\tilde{p} - \tilde{q} = \vec{v}$$

Conversely if i start with a point, and tell you to move by some vector, i will get to another point

$$\tilde{q} + \vec{v} = \tilde{p}$$

1.1.1 Frames

In the previous section we discussed how a basis was made up of three vectors, and one could describe any arbitrary vector as a linear combination of of these vectors.

$$\vec{v} = \sum_i c_i \vec{b}_i$$

How do we extend this setting to the using coordinates vectors with an affine space of points? What plays the role of a basis?

In an affine space, our plan is to describe any point $\tilde{p}$ by first starting from some origin point $\tilde{o}$, and then adding to it a linear combination of vectors using coordinates c_i to weight three basis vectors $\vec{b}_i$.

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b} \quad = \quad \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} = \vec{\mathbf{f}}^t \mathbf{c}$$

The row

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} = \vec{\mathbf{f}}^t$$

is called an affine frame for the space A^3. It is like a basis, but it is made up of three vectors and a single origin point.

In order to specify a point using a frame, we need to use a homogeneous coordinate vector. That is a vector with four numbers, with the last one always being a one.

What happens if have a frame and use a homogeneous coordintate vector with a zero as the last element, we get

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \end{bmatrix} = \vec{v}$$

This is just a linear combination of basis vectors, with no contribution from the origin point, and so we obtain a vector.

1.2 Homogeneous Coordinates and affine combinations

To describe a point, given some frame, we use four coordinates. When we wish to represent a point, we set the last coordinate to 1.

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

When we wish to represent a vector, we set the last coordinate to 0

$$\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

This convention works perfectly with the operations we discussed above. If I subtract the coordinates of two points using coordinate vector algebra, I will get a set of coordinates of a vector.

$$\begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix} - \begin{bmatrix} d \\ e \\ f \\ 1 \end{bmatrix} = \begin{bmatrix} a-d \\ b-e \\ c-f \\ 0 \end{bmatrix}$$

If I add a vector to a point, I get a vector. If I add two vectors I get a vector. If I multiply a scalar by a vector I get a vector. If I do anything else, I will generally get some non zero or one last coordinate, so it is an undefined operations.

Suppose i attempt to scalar two points by scalar weights α_i and then add them together, then the forth coordinate will generally be non zero or one. But lets look at what happens when $\alpha_0 + \alpha_1 = 1$. In this case, the forth coordinate will be a 1

$$\alpha_0 \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} d \\ e \\ f \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_0 a + \alpha_1 d \\ \alpha_0 b + \alpha_1 e \\ \alpha_0 c + \alpha_1 f \\ \alpha_0 + \alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 a + \alpha_1 d \\ \alpha_0 b + \alpha_1 e \\ \alpha_0 c + \alpha_1 f \\ 1 \end{bmatrix}$$

It appears that we have described a valid point. This actually does make some kind of sence. If, for example, i add $1/2$ of the first point to $1/2$ of the second point, i am actually describing a new "average" point halfway inbetween them. It turns out, that we haven't even broken our earlier rules about the data types.

$$\begin{aligned} \alpha_1 \tilde{p_1} + \alpha_2 \tilde{p_2} &= \\ (1-\alpha_2)\tilde{p_1} + \alpha_2 \tilde{p_2} &= \\ \tilde{p_1} + \alpha_2(\tilde{p_2} - \tilde{p_1}) & \end{aligned}$$

This operation is simply built up from our valid point/vector operations. This kind of operation is so useful that we give it a new name, an affine combination of points. We can also genearalize this operation and define affine combinations of n points

$$\sum_i \alpha_i \tilde{p_i} \quad \text{where} \sum_i \alpha_i = 1$$

An affine combination of two points will give me some point on the line spanned by the two points. If $0 < \alpha_i < 1$, then we call this a convex combination, and the resulting point is on the segment between the two points. If some of the α_i are out of this range, then the resulting point is on the line, but is outside of the segement.

An affine combination of three points will give me some point on the plane spanned by the three points. If the affine combination is also convex, then the resulting point will be in the interior of the triangle defined by the three points.

1.3 Affine transformations and Four by four matrices

Lets look at what happens when we muliply a homogeneous coordinate vector by a 4 by 4 matrix of the form

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We call such a matrix, an affine matrix.

If the coordinate vector has a 1 as the last coordinate, we will obtaint a coordnate vector for a new point.

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

If the coordinate vector has a 0 as the last coordinate, we will obtaint a coordnate vector for a new vector.

$$\begin{bmatrix} x' \\ y' \\ z' \\ 0 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

So 4 by 4 matrices with a last row of $[0, 0, 0, 1]$ can be used to map points to points and vectors to vectors. If the last row were anything else, it would generally give us an invalid result.

1.3.1 Using a 4 by 4 matrix

In a linear space, we applied a linear transformation to a vector by placing a matrix between the basis and the coordinate vector. We perform an affine transformation on a point by placing an affine matrix, a matrix with last row $[0, 0, 0, 1]$, between a frame and a homogeneous coordinate vector.

We can read this both left to right, or right to left as we did with linear transformations.

$$
\begin{aligned}
& \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \\
\Rightarrow & \left(\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \\
= & \begin{bmatrix} \vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3 & \tilde{o}' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}
\end{aligned}
$$

When read left to right, we see the transformation doing its job by changing the frame, but sticking with the same homogeneous coordinate vector.

Here we see another reason why the last row has to be $[0, 0, 0, 1]$. Each of the first three columns of the matrix describes how three new frame vectors $\vec{b_i}'$ are obtained for the frame, by linearly combining the three original frame vectors $\vec{b_i}$. The last column describes how the new origin point $\tilde{o}'$ is obtained by starting from the original origin point $\tilde{o}$, and adding to it some combination of the frame vectors.

Of course since matrix multiplication is associative, we can interpret the affine from right to left.

$$
\begin{aligned}
& \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \\
\Rightarrow & \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \left(\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \right) \\
= & \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c'_1 \\ c'_2 \\ c'_3 \\ 1 \end{bmatrix}
\end{aligned}
$$

Under this interpretation, the affine transformation does its job by chaning the homogeneous coordinate vector which is then used with the original frame.

1.4 Examples of affine transformations rotations and translations

A 3D translation by $[t_x, t_y, t_z]^t$ performs the following transformation

$$\begin{aligned} x' &= x + t_x \\ y' &= y + t_y \\ z' &= z + t_z \end{aligned}$$

Using homogeneous coordinates, we see that a point can be translated using

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

What happens if we apply a translation matrix to a point

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

We obtain a new tranlsated point.

What happens if we apply a translation matrix to a vector

$$\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

We obtain the original vector. So we see that a translation does not effect vectors, only points. This makes sence since a vector represents motion, for example a step in the x direction. This step is the same whether I start the step in boston or seattle.

Just like we could rotate vectors using a 3 by 3 matrix, we can rotate points about the origin of a frame using a matrix of the form

$$R = \begin{bmatrix} k_x^2 v + c & k_x k_y v - k_z s & k_x k_z v + k_y s & 0 \\ k_y k_x v + k_z s & k_y^2 v + c & k_y k_z v - k_x s & 0 \\ k_z k_x v - k_y s & k_z k_y v + k_x s & k_z^2 v + c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If we multiply a translation and a rotation matrix together: we get

$$TR = \begin{bmatrix} k_x^2 v + c & k_x k_y v - k_z s & k_x k_z v + k_y s & t_x \\ k_y k_x v + k_z s & k_y^2 v + c & k_y k_z v - k_x s & t_y \\ k_z k_x v - k_y s & k_z k_y v + k_x s & k_z^2 v + c & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This means that if someone give you a matrix of the form above, you can easily factor it into $M = TR$.

In fact this is almost all there is to rigid transformations, since if i compose any sequence of rotations and translations, i will always get some matrix with the above form, which can always be decomposed into the form $M = TR$

Lets see why this is. First of all any succesive sequence of rotations can be collapesed into a single rotation $R_1 R_2 ... R_n = R$. Second of all, any successive sequence of translations can be collaped into a single translation $T_1 T_2 ... T_n = T$. Finally, traslations can easily be moved from the right of a rotation matrix to its left. $RT' = RT'(R^{-1}R) = (RT'R^{-1})R = TR$. The first step is done by mulitplying in R together with its inverse. The second step is to rearange the parenthesis. The third step is discover that the matrix $RT'R^{-1}$ is actually a different translation matrix T. If T' is the matrix

$$\begin{bmatrix} 1 & 0 & 0 & t'_x \\ 0 & 1 & 0 & t'_y \\ 0 & 0 & 1 & t'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then the new matrix T is

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$\begin{bmatrix} t_x \\ t_y \\ t_z \\ 0 \end{bmatrix} = R \begin{bmatrix} t'_x \\ t'_y \\ t'_z \\ 0 \end{bmatrix}$$

1.5 Multiple Transformations

We will use this “left of transformation matrix” rule to interpret sequences of multiple transformations. We will do this by looking at a specific example.

In the following 2D example, we use a rotation matrix R and a translation matrix T, The rotation matrix will have the effect of rotating a point by θ degrees about the current coordinate system origin. The translation matrix will have the effect of translating the point by one unit in the direction of the first basis element axis.

We will now interpret the following transformation

$$\vec{\mathbf{f}}^t \mathbf{c} \Rightarrow \vec{\mathbf{f}}^t RT \mathbf{c}$$

We will do this by breaking up the transformation into two steps. In the first step we will transform the point only using the R matrix.

$$\vec{\mathbf{f}}^t\mathbf{c} \Rightarrow \vec{\mathbf{f}}^t R\mathbf{c} = \vec{\mathbf{f}'}^t\mathbf{c}$$

In this fist step, the point $\tilde{p}$ is rotated; and this rotation is done about the frame $\vec{\mathbf{f}}^t$. The resulting point can also be expressed as $\vec{\mathbf{f}'}^t\mathbf{c}$.

In the second step, this resulting point is now transformed using T

$$\vec{\mathbf{f}}^t R\mathbf{c} \Rightarrow \vec{\mathbf{f}}^t RT\mathbf{c}$$

which is the same as

$$\vec{\mathbf{f}'}^t\mathbf{c} \Rightarrow \vec{\mathbf{f}'}^t T\mathbf{c}$$

In this second step the matrix T was placed with the basis $\vec{\mathbf{f}'}^t$ immediately to its left. As such the translation is done along the direction of the first basis axis of $\vec{\mathbf{f}'}^t$.

We can also interpret this transformation in another valid way. This will be done by applying the rotation and translation in the other order.

In the first step, we apply the transformation

$$\vec{\mathbf{f}}^{\mathbf{t}}\mathbf{c} \Rightarrow \vec{\mathbf{f}}^{\mathbf{t}}\mathbf{T}\mathbf{c} = \vec{\mathbf{f}}^{\mathbf{t}}\mathbf{c}'$$

The point is first translated one unit in the direction of the first basis axis of $\vec{\mathbf{f}}^t$. The new point can be expressed as $\vec{\mathbf{f}}^t\mathbf{c}'$ for the appropriate homogeneous coodinate vector $\mathbf{c}' = T\mathbf{c}$. In the second step we transform the resulting point using the rotation matrix R.

$$\vec{\mathbf{f}}^{\mathbf{t}}\mathbf{T}\mathbf{c} \Rightarrow \vec{\mathbf{f}}^{\mathbf{t}}\mathbf{R}\mathbf{T}\mathbf{c}$$

This is the same as

$$\vec{\mathbf{f}}^{\mathbf{t}}\mathbf{c}' \Rightarrow \vec{\mathbf{f}}^{\mathbf{t}}\mathbf{R}\mathbf{c}'$$

In this transformation R is placed with $\vec{\mathbf{f}}^t$ immediately to its left, and so the rotation of the point is done with respect to $\vec{\mathbf{f}}^t$.

These are just two different interpretations of the original transformation sequence. 1) Rotate point with repsect to $\vec{\mathbf{f}}^t$ then translate that point with respect to $\vec{\mathbf{f}'}^t = \vec{\mathbf{f}}^t R$. 2) Translate the point with respect to $\vec{\mathbf{f}}^t$ then rotate that resulting point with respect to the frame $\vec{\mathbf{f}}^t$ At times it will be more convenient to use the first interpretation and at other times it may be more convenient to use the second one.

1.5.1 Multiple Transformations to a frame

In the same way that we had dual interpretations for the RT sequence applied to a point, we have can interpret transformations applied to frames in two ways as well.

Given the basis transformation

$$\vec{\mathbf{f}}^{\mathbf{t}} \Rightarrow \vec{\mathbf{f}}^{\mathbf{t}}\mathbf{R}\mathbf{T}$$

we can read this as

$$\vec{\mathbf{f}}^{\mathbf{t}} \Rightarrow \vec{\mathbf{f}}^{\mathbf{t}}\mathbf{R} \Rightarrow \vec{\mathbf{f}}^{\mathbf{t}}\mathbf{R}\mathbf{T}$$

We would interpret this as first rotating the frame $\vec{\mathbf{f}}^{t}$ with respect to $\vec{\mathbf{f}}^{t}$ to obtain a new frame $\vec{\mathbf{f}'}^{t}$. The new frame is then translated with respect to $\vec{\mathbf{f}'}^{t}$ to obtain the final frame.

We can also apply the transformations in the other order,

$$\vec{\mathbf{f}}^{\mathbf{t}} \Rightarrow \vec{\mathbf{f}}^{\mathbf{t}}\mathbf{T} \Rightarrow \vec{\mathbf{f}}^{\mathbf{t}}\mathbf{R}\mathbf{T}$$

In this case $\vec{\mathbf{f}}^{t}$ is first translated with respect to $\vec{\mathbf{f}}^{t}$ to obtain a new frame. This resulting frame is then rotated with respect to $\vec{\mathbf{f}}^{t}$ to obtain the final frame.

These types of interpretations are often summarized as follows: If one read transformations from left to right, then one each transform is done with respect to a newly created "local" frame. If one reads the transformations from right left, then each transform is done with respec to the original "global" frame.

1.6 Another use of matrices: Expressing same point in different frame

A matrix can be used to describe the transformation of a point to another point $\tilde{p} \Rightarrow \tilde{p'}$. A matrix can also be used to express a single point $\tilde{p}$ with respect to two different frames $\vec{\mathbf{f}}^{t}$ and $\vec{\mathbf{a}}^{t}$.

Using the first frame, the point can be expressed using some appropriate homogeneous coordinate vector $\tilde{p} = \vec{\mathbf{f}}^{t}\mathbf{c}$.

Suppose that my two frames are related by the following matrix equation $\vec{\mathbf{a}}^{\mathbf{t}} = \vec{\mathbf{f}}^{\mathbf{t}}\mathbf{M}$ where M is some four by four matrix. Then i can write $\tilde{p}$ as

$$\tilde{p} = \vec{\mathbf{f}}^{t}\mathbf{c} = \vec{\mathbf{a}}^{t}M^{-1}\mathbf{c} = \vec{\mathbf{a}}^{t}\mathbf{d}$$

This is not a transformation (the $\Rightarrow$ notation), but an equation (the $=$ notation). We have simply written the same point using two bases.

$$\mathbf{d} = M^{-1}\mathbf{c}$$

1.6.1 Transforms using an auxiliary coordinate system

There are many times when one wishes to transform a frame $\vec{\mathbf{f}}^{t}$ in some specific way, lets say a rotation R, with respect to some auxiliary frame $\vec{\mathbf{a}}^{t}$. For example i may want to rotate the geometry representing a small child about a frame representing a merry-go-round.

This is easy to do as long as i know the matrix relating $\vec{\mathbf{f}}^t$ and $\vec{\mathbf{a}}^t$. For example we may know that

$$\vec{\mathbf{f}}^{\mathbf{t}}\mathbf{M} = \vec{\mathbf{a}}^{\mathbf{t}}$$

The transformed coordinate system can then be expressed as

$$\begin{aligned} & \vec{\mathbf{f}}^{\mathbf{t}} \\ = \quad & \vec{\mathbf{a}}^{\mathbf{t}}\mathbf{M}^{-1} \\ \Rightarrow \quad & \vec{\mathbf{a}}^{\mathbf{t}}\mathbf{R}\mathbf{M}^{-1} \\ = \quad & \vec{\mathbf{f}}^{\mathbf{t}}\mathbf{M}\mathbf{R}\mathbf{M}^{-1} \end{aligned}$$

In the first line, we rewrote the frame $\vec{\mathbf{f}}^t$ using $\vec{\mathbf{a}}^t$. In the second line we transformed the frame system using the “left of” rule. We transformed rotated our frame using R with respect to $\vec{\mathbf{a}}^t$. In the final line, we simply rewrote the expression to remove the auxiliary frame.